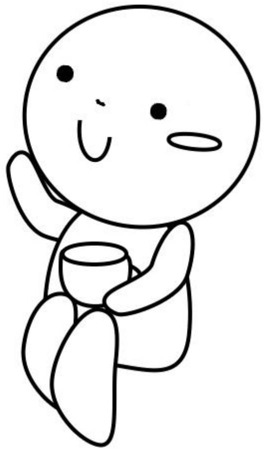
目錄



編者的話

相信同學都有不少「解難」的經歷，在解題的過程中品嚐無盡的苦與樂，不知大家有否領略在多變的解題方法中滲透出無限的精彩呢? 其實從題目所涉及的知識層面或其所呈現的結構特徵，很多數學命題和定理，都能用一種或以上的方法去證明，我們把這簡稱為「一題多解」。

適逢屯天廿五週年銀禧校慶，本期《數味軒》以「一題多解」為主題，輯錄 25 道問題，提供多個解答方法，讓讀者通過一題多思，一題多解，一題多講，從中欣賞到數學多變而有趣的一面；利用多角度看一道題，不但強化思維的連貫性，更重視知識的銜接。讀者不妨動手做一做，再看看自己解題的方向與輯錄的方法有何異同。通過不斷嘗試，讀者既可活用知識，也能感受到箇中的奧妙之處。此外，本期的封底是一個數字連線遊戲，讀者可按數字順序連線，看看得到什麼的圖案。

在此特別鳴謝一眾工作人員，尤其是勞苦功高的潘雪芬老師，以及七位校友的全力協助，包括撰寫序言的蕭金源、李家利和莫家榮；校對題目的梁鎮浩；搜集資料的曾金標和蔡浩賢；還有為數味軒打造全新封面設計的陸健輝。沒有他們的努力，本期厚達五十多頁的數味軒不能這樣順利地誕生。

希望各位讀者能在「一題多解」中感受到數學美妙的一面，活學活用不同的數學概念，並從中建立多角度思維。如有任何感想或意見，歡迎向顧問老師或數學學會提出。

5A 劉靜鴻

11–12 數學學會主席

序（一）

「條條大路通羅馬」，相信大家對這句話已經是耳熟能詳，將這句話套進數學解題中，即代表每道題目都可以使用多於一種解題方法找出答案。今期《數味軒》的主題——「一題多解」，充分表現其無窮的魅力。

究竟「一題多解」有何好處？我覺得最主要的好處是鍛鍊自身對數學的審美觀。以當中的一道題目為例："Prove that $23^n - 19^n$ is divisible by 4 for all positive integer *n*."，數學歸納法及二項式定理都是解題方法，但哪一個較佳？我們明白數學歸納法是要知道題目的結果才可以論證，至於另一方法美妙之處就是把 23 改寫成 $19+4$，然後再配合二項式定理，便可以得到結果。由此可見二項式定理比數學歸納法較佳，而我們對數學的審美觀也從中得到鍛鍊。

「一題多解」另一個好處是熟悉且掌握問題背後所涉及的各種數學概念，從而明白每一個解題方法有什麼限制及其應用層面。笛卡兒曾說：「從求解一個問題中，我就可以形成一個方法，以備往後求解其他問題之用。」透過「一題多解」，我們會知道有些方法比較淺顯，應用有限；有些方法較深刻，並且應用廣泛。這樣，當我們遇到不同類型的題目時，我們知道應該使用哪一種方法能有效解題。

另一方面，我想問大家有否堅持過「一題多解」？堅持「一題多解」可以令我們意識到每道數學題目都應有更好的解題方法，這可驅使自身不斷自我完善。千萬不要小看這力量，很多數學家都是因此而想到很多不同的數學概念。究竟這種力量有多麼的驚人，就留待大家去發掘。

蕭金源
2010 – 2011 7B
數學學會主席 (2009 –2010)
香港中文大學 計算機工程學系
一年級學生(2011)

序（二）

有時候，一道數學題，答案並不是最重要的，其解答過程反倒是精髓所在。為什麼這麼說呢？相信大家只要翻開今期的《數味軒》就能體會到了！聰明的你也許早就猜到，這次數味軒的主題正是一題多解。

談到一題多解，不由得想起小學時代——那時候我最喜歡的科目是數學。記得當時那位年輕的數學老師在課堂上總愛叫同學出來把問題的答案寫在黑板上，答對了可以獲得一個小禮物，於是大家總是很踴躍地舉手。無奈僧多粥少，能被老師點名出去表現自己的只有一位 —— 除非主動跟老師提議說自己有另外一個解法，那麼就可以出去把自己的答案寫在黑板上了。從那時起，我便漸漸養成了盡量思考不同的方法去解答同一條題目的習慣。後來才發現，這種一題多解的思考方法，不僅鍛煉了我的解題能力，還讓我體會到思考的樂趣，更埋下了日後修讀理科的種子。

除了可以鍛煉思考，一題多解還能啟發人的創新能力。如果僅僅滿足於答案的獲得，那麼人的進步空間將會十分有限；相反，一題多解的精神則可以使人在知道問題的答案後，仍然保持思索與探究的熱忱，務求找到更佳的解決方法。例如，要寫字的話，用古時候的毛筆就可以解決，可如果人們僅僅滿足於此，便不會出現更為方便耐用的圓珠筆了— 許多偉大的靈感與創造就是在這尋求一題多解的過程中誕生的。

希望同學們可以在我們精心挑選的 25 道示範題中，找到一題多解的意義與樂趣，並找機會在日後的數學學習及日常生活中應用一題多解的神奇魔力，說不定你也能成為一名改變世界的發明家！

李家利

2010 – 2011 7B
香港科技大學
環境管理及科技學系
一年級學生(2011)

序（三）

初知道自己要為母校《數味軒》撰寫序時，我突然感到不知所措，畢竟自己在預科生活中已與數學這一門學科扯不上任何關係。回想中學五年學習數學歷程，我偶爾會問自己一些問題，相信這亦是一眾同學心中的疑惑 — 四則運算以外的數學理論與我何干？沒有了數學就不能生活嗎？到了高年級，心中的疑惑就會全消。Carl Friedrich Gauss 曾說，數學是科學之母。無論物理化學等科目，都需要運用到微積分等高階數學概念。若果沒有了數學，恐怕大家也不能用電腦瀏覽社交網站了。沒有了數學，我們還可以享受如此便利的生活嗎？

《數味軒》在屯天已經出版了多個年頭。適逢銀禧紀念，數學學會呼應學校「創新」口號，一改以往每期搜集數篇與數學相關的文章而加以討論；今年則以主題式的出版，數學學會及其顧問老師共提供了二十五題數學趣味題予同學思考，藉以訓練和挑戰自己。有人認為，數學是一門扼殺同學創意的學科。然而，縱然題目的正確答案只得一個，解題方法卻是無窮無盡，給予同學極廣闊的思考空間。這亦是今年《數味軒》的一個發展方向 — 通過「一題多解」令同學明白數學是多元有趣的科目。

閱讀《數味軒》並非單單只是高年級同學的專利。對於低年級同學而言，你們亦可嘗試解答當中題目。「一題多解」之妙，就是可以讓同學利用不同數學理論解題。例如在一條有關於畢氏定理的題目中，解題方法就有多種。低年級同學可嘗試充分利用所學所知，測試自己的實力，待自己到了更高年級，重新思考題目一次，看自己能否領會更多。至於高年級的同學，千萬不要想到一個解題方法後就沾沾自喜，不妨多花時間挑戰每一條題目，以了解自己能否靈活運用多年數學知識，得出不同解題方法。今年彩色印刷的《數味軒》，相信定必能令同學更清晰閱讀題目，讀起來更賞心悅目。

雖然我對數學的興趣不很濃厚，但閱畢《數味軒》後，卻重拾了初中時

成功解題的滿足感，覺得數學是有趣的學科。希望熱衷數學的同學，能在此書進一步發掘數學有趣之處；不太喜歡數學的同學，可以對這一門學科改觀，有另一番體會。我深信，此書必能對訓練同學的思維有莫大裨益。

莫家榮

2010 – 2011　7A

香港城市大學法律系

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一題多解

題目一

$1 + 3 + 5 + 7 + \ldots + 89 = ?$

方法一

$$1 = 1 \times 1$$
$$1 + 3 = 2 \times 2$$
$$1 + 3 + 5 = 3 \times 3$$
$$\vdots$$
$$1 + 3 + 5 + 7 + \ldots + 89 = 45 \times 45$$

因此，求 1 + 3 + 5 + 7 +…+ 89 的值，等同計算邊長為 45 的正方形的面積。

方法二

設 $S = 1 + 3 + 5 + 7 + \ldots + 89 \quad \ldots (1)$

$S = 89 + 87 + 85 + 83 + \ldots + 1 \quad \ldots (2)$

$(1) + (2) \quad 2S = 90 \times 45$

$S = 45^2$

方法三

$$1 = 1 \times 1 \times 1 = 1^3$$
$$3 + 5 = 2 \times 2 \times 2 = 2^3$$
$$7 + 9 + 11 = 3 \times 3 \times 3 = 3^3$$
$$\vdots$$
$$73 + 75 + \ldots + 89 = 9 \times 9 \times 9 = 9^3$$

$\therefore \quad 1 + 3 + 5 + \ldots + 89 = 1^3 + 2^3 + 3^3 + \ldots + 9^3$

$$1^3 = 1^2$$
$$1^3 + 2^3 = 9 = (1 + 2)^2$$
$$1^3 + 2^3 + 3^3 = 36 = (1 + 2 + 3)^2$$
$$\vdots$$

$\therefore 1^3 + 2^3 + 3^3 + \ldots + 9^3 = (1 + 2 + 3 + \cdots + 9)^2 = 45^2$

方法四

$$1 + 3 + 5 + 7 + \ldots + 89 = \sum_{i=1}^{45}(2i-1) = 2\sum_{i=1}^{45}(i) - 45 = 45(46) - 45 = 45^2$$

題目二

求 1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 + …+ 2005 + 2006 – 2007 – 2008 + 2009 + 2010 – 2011 – 2012 的值。

方法一

1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 + … + 2005 + 2006 – 2007 – 2008 + 2009 + 2010 – 2011 – 2012

= 1 + (2 – 3 – 4 + 5) + (6 – 7 – 8 + 9) + (10 – 11 – 12 + 13) + … + (2006 – 2007 – 2008 + 2009) + (2010 – 2011) – 2012

= 1 + 0 + 0 + 0 +… + (–1) – 2012

= –2012

方法二

1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 + … + 2005 + 2006 – 2007 – 2008 + 2009 + 2010 – 2011 – 2012

= (–0 + 1 + 2 – 3) + (– 4 + 5 + 6 – 7) + (– 8 + 9 + 10 – 11) + … + (– 2008 + 2009 + 2010 – 2011) – 2012

= 0 + 0 + 0 +… + 0 – 2012

= – 2012

方法三

1 + 2 – 3 – 4 = –4

1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 = – 8

1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 = –12

⋮

1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 + … + 2005 + 2006 – 2007 – 2008 + 2009 + 2010 – 2011 – 2012

= –2012

方法四

隔數相加：如 1 – 3 = –2，2 – 4 = –2，5 – 7 = –2，…，這樣的數對共有 1006 對，

∴原式 = –2 × 1006 = –2012.

方法五

設 $z=\sum_{r=1}^{2012} ri^{r-1}$，當中 $i^2=-1$.

1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 + …+ 2005 + 2006 – 2007 – 2008 + 2009 + 2010 – 2011 – 2012 = Re (z) + Im (z).

$$\sum_{r=1}^{n} x^r = \frac{x^{n+1}-1}{x-1}$$

$$\frac{d}{dx}\left(\sum_{r=1}^{n} x^r\right) = \frac{d}{dx}\left(\frac{x^{n+1}-1}{x-1}\right)$$

$$\sum_{r=1}^{n} rx^{r-1} = \frac{nx^{n+1}-(n+1)x^n+1}{\left(x-1\right)^2} \ldots (*)$$

$\therefore$將 n = 2012 和 $x = i$, 代入(*)，

$$z = \; = \frac{2012\left(i^{2013}\right)-2013\left(i^{2012}\right)+1}{\left(i-1\right)^2} = -1006-1006i$$

$\therefore$ 1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + 9 + 10 – 11 – 12 + …+ 2005 + 2006 – 2007 – 2008 + 2009 + 2010 – 2011 – 2012 = –2012

題目三　畢氏定理

圖 1 中，ΔPQR 為一個直角三角形，

$PQ = a$，$PR = b$ 和 $QR = c$。

由此得知　$a^2 + b^2 = c^2$。

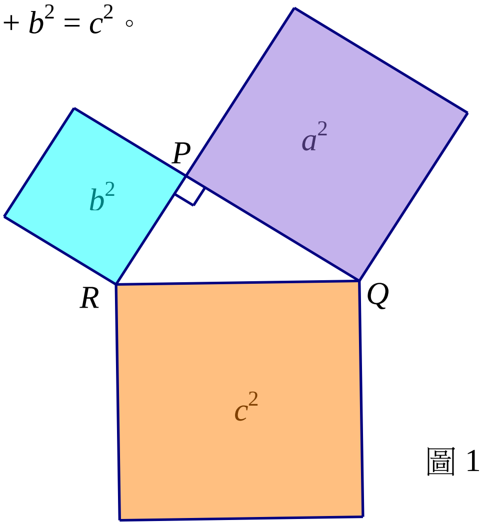
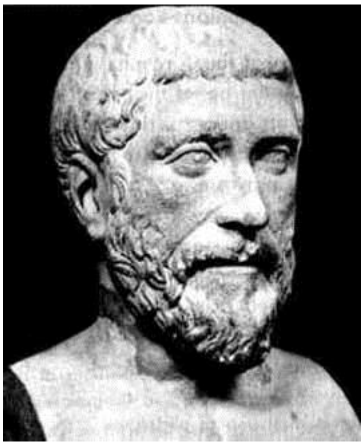


圖 1



畢達哥拉斯

(570 BC - 475 BC)

證明一 (出入相補證明)

圖 1.1 中，正方形 $ABCD$、$AXYZ$ 和 $MYNC$ 的邊長分別為 a、b 和 c。

從切割的拼圖得知　$a^2 + b^2 = c^2$。

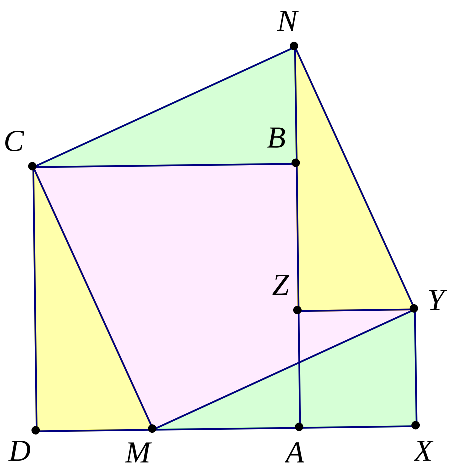
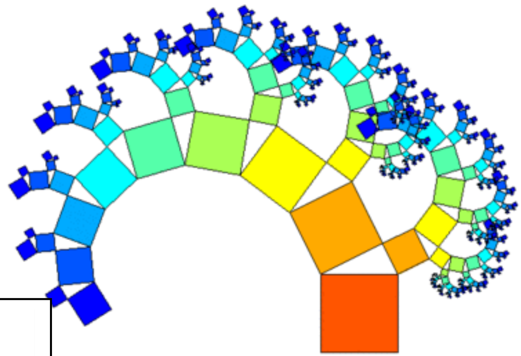


圖 1.1



Source: **Pythagoras** tree 8 jet.gif

commons.wikimedia.org

證明二

將四個圖 1 中的直角三角形拼合成正方形 *ABCD*，如圖 1.2 所示。

正方形 *ABCD* 面積 = 4×(直角三角形面積) + 正方形 EFGH 面積

$$(a+b)^2 = 4\times\left(\frac{1}{2}ab\right) + c^2$$

$$a^2 + b^2 = c^2$$

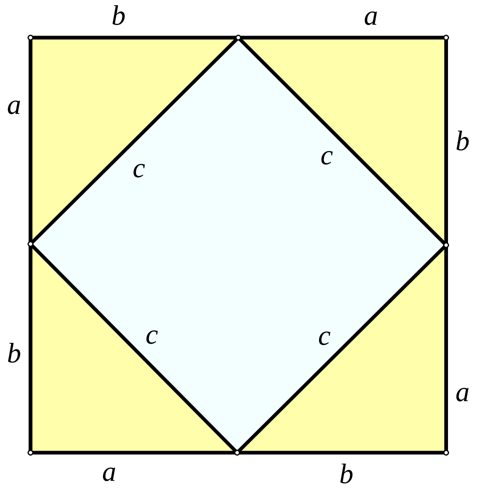


圖 1.2

證明三

將四個圖 1 中的直角三角形拼合成正方形 *ABCD*，如圖 1.3 所示。

正方形 *ABCD* 面積 = 4×(直角三角形面積) + 正方形 *EFGH* 面積

$$c^2 = 4\times\left(\frac{1}{2}ab\right) + (b-a)^2$$

$$a^2 + b^2 = c^2$$

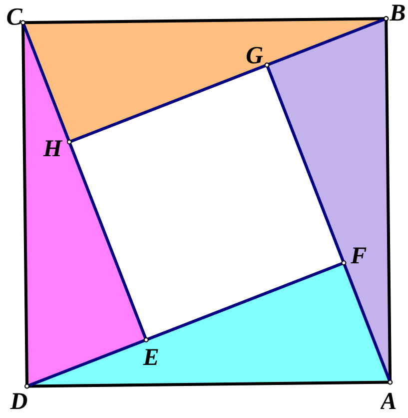


圖 1.3

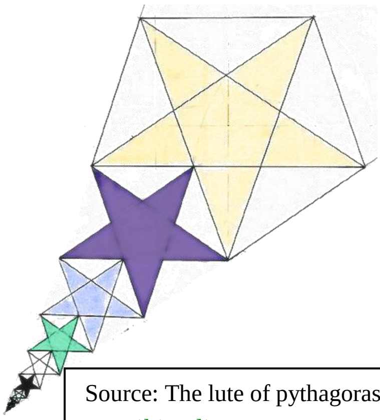
其他畢氏定理的證明，同學可參考以下網址：

http://www.cut-the-knot.com/pythagoras/

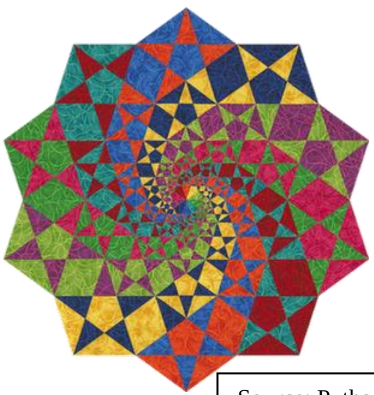
http://www.math.ubc.ca/~cass/courses/java/m308/pythagoras.html

http://www.jes.co.jp/math/java/pythagoras.html

http://mathforum.com/~isaac/problems/pythagthm.html



Source: The lute of pythagoras.jpg.
en.wikipedia.org



Source: Pythagoras' Lute Quilt Pattern is based on Pythagoras' Golden Triangle,
stephenseifert.blogspot.com

題目四

圖 2 中，ΔABC 是一個等腰直角三角形，$AC = BC$，且 $AB = 10$ cm，求ΔABC 的面積。

方法一

設 $BC = AC = x$ cm。

跟據畢氏定理，$x^2 + x^2 = 10^2$

$x^2 = 50$

$\therefore \Delta ABC$ 的面積 $= \frac{1}{2}x^2 = 25\ (\text{cm}^2)$。

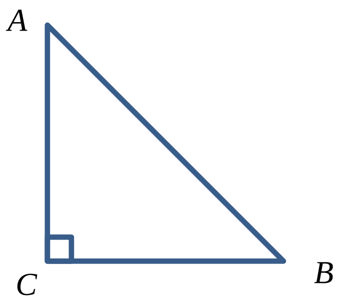


圖 2

方法二

作斜邊 AB 上的高 CD，如圖 2.1 所示。

此時ΔBCD 和ΔACD 皆為等腰直角三角形，

故 $AD = CD = BD = \frac{1}{2}AB = 5$ cm，

$\therefore \Delta ABC$ 的面積 $= \frac{1}{2} \times 10 \times 5 = 25\ (\text{cm}^2)$。

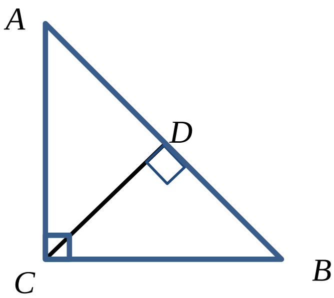


圖 2.1

方法三

將 AC 與 BC 各延長一倍，使 $BC = CD$ 和 $AC = CE$，如圖 2.2 所示。

連接 AD、DE 和 BE。

此時 $ABED$ 為一個正方形，且面積是 10^2 cm^2。

而ΔABC 的面積是正方形 $ABED$ 面積的四分之一，所以

ΔABC 的面積 $= \frac{1}{4} \times 10 \times 10 = 25\ (\text{cm}^2)$。

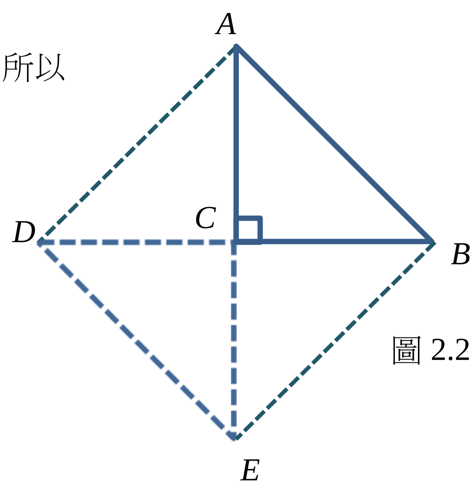


圖 2.2

題目五

圖 3 中，正方形 $ABCD$ 的邊長為 4cm，在其內部置入兩個全等圓，此二圓互相外切且又與正方形的兩邊相切，求圓之半徑。

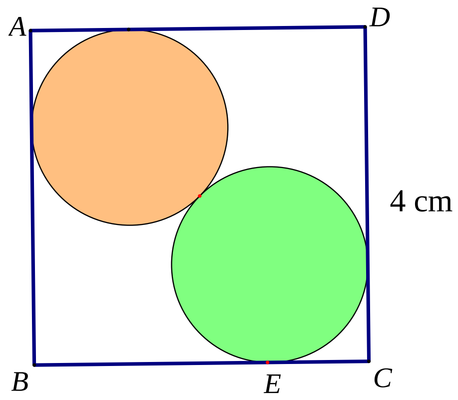


圖 3

方法一

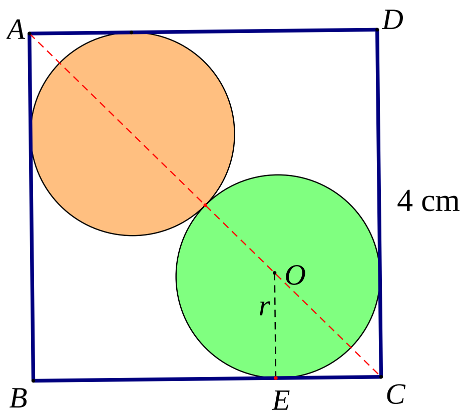


圖 3.1

如圖 3.1 所示，設 O 為圓心，半徑為 r cm。

ΔOEC 為等腰直角三角形，故 $OE = EC = r$ cm，

而 $OC = \sqrt{2}\,r$ cm，

$AC = 2\sqrt{2}\,r + 2r = 2r\left(\sqrt{2}+1\right)$ (cm)。

$\because \Delta ABC$ 為等腰直角三角形

$\therefore AC = \sqrt{4^2+4^2} = 4\sqrt{2}$ cm

$2r\left(\sqrt{2}+1\right) = 4\sqrt{2}$

$$r = \frac{2\sqrt{2}}{\sqrt{2}+1} = \frac{2\sqrt{2}\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} = 4-2\sqrt{2}$$

$\therefore$ 圓半徑為 $\left(4-2\sqrt{2}\right)$cm。

方法二

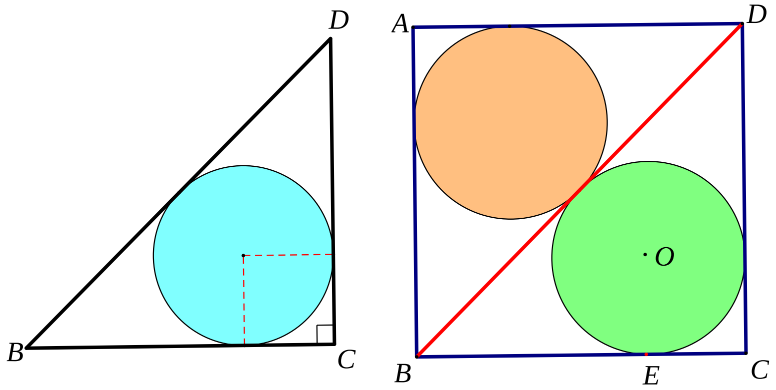


圖 3.2

依據內切圓的概念，內切圓的半徑 $= \dfrac{BC + CD - BD}{2}$。

如圖 3.2 所示，連接 BD，此時以 O 為圓心的圓即為ΔBCD 的內切圓，

$\therefore$ 內切圓的半徑 $= \dfrac{BC + CD - BD}{2} = \dfrac{4+4-4\sqrt{2}}{2} = \left(4-2\sqrt{2}\right)$cm。

方法三

設 O 與 Y 分別為二圓的圓心，半徑為 r cm。

如圖 3.3 所示，連接 OX、XY、YZ 和 OZ，

$OXYZ$ 為一正方形，$YZ = OZ = (4 - 2r)$ (cm)。

在 ΔOYZ，$YZ^2 + OZ^2 = OY^2$

$$2(4-2r)^2 = (2r)^2$$
$$4-2r = \sqrt{2}r$$
$$r = \frac{4}{\sqrt{2}+2}$$
$$= \frac{4}{\sqrt{2}+2} \times \frac{\left(2-\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}$$
$$= 4-2\sqrt{2}$$

$\therefore$ 圓半徑為$\left(4-2\sqrt{2}\right)$cm。

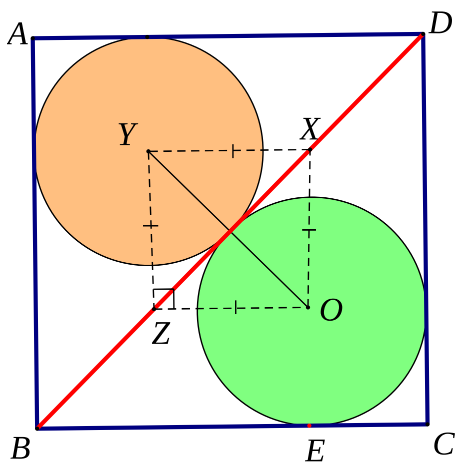


圖 3.3

題目六

一個由邊長 2cm 的正十二邊形被割走 12 個邊長 2cm 的等邊三角形後而成的星形圖案，求星形圖案面積。

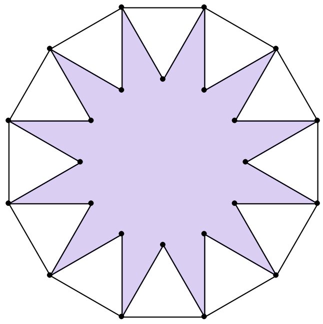


圖 4

方法一

如圖 4.1，把原圖中央部分分為 6 等份。

$\angle CED = 360° \div 6 = 60°$

$\because ED = EC$

$\therefore \angle CDE = \angle ECD$ (等腰三角形底角相等)

$\angle CDE = (180° - 60°) \div 2 = 60°$ (三角形的內角和)

$\because \angle CED = \angle CDE = \angle ECD = 60°$

$\therefore \Delta CDE$ 是個等邊三角形

十二邊形內角 $= (12 - 2) \times 180° \div 12 = 150°$

(多邊形的內角和)

$\angle BAD = \angle ABC = 150° - 60° = 90°$

$\angle ADC = \angle BCD = [(4-2) \times 180° - 90° \times 2] \div 2 = 90°$

(多邊形的內角和)

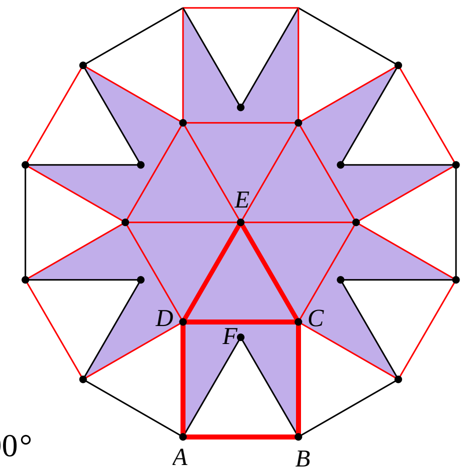


圖 4.1

$\because AB = AD$ 及 $\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90°$

$\therefore ABCD$ 為一個正方形

$\because CD = AB$，且 ΔCDE 及 ΔABF 均為等邊三角形

$\therefore \Delta CDE$ 和 ΔABF 為全等三角形

所以 $AFBCED$ 的面積等於正方形 $ABCD$ 的面積$=2\text{cm} \times 2\text{cm} = 4\text{ cm}^2$

所以全個圖形面積為 $2\text{cm} \times 2\text{cm} \times 6\text{cm} = 24\text{ cm}^2$。

方法二

如圖 4.2，以 O 為中心把圖分為 12 等份，OAD 為一直線且垂直於 BC。

$\angle BOC = 360°\div 12 = 30°$ (同頂角)

跟據對稱性，$\angle BAD = \frac{1}{2}\angle BAC = 30°$

$\angle BOA = \angle COA = 15°$

$\angle OBA = 15°$ (三角形的外角)

$\therefore\ \Delta BOA$ 為等腰三角形，且 $OA = AB = 2\text{cm}$

在 ΔABC，$\tan 60° = AD \div BD$

$AD = \tan 60° \times BD = \sqrt{3}\ \text{cm}$

$\therefore\ OD = (2+\sqrt{3})\ \text{cm}$

$\therefore\ OBAC$ 的面積 $= \Delta OBC$ 與 ΔABC 的面積之差

$= \left[\left(2+\sqrt{3}\right)\times 2\div 2-\left(\sqrt{3}\right)\times 2\div 2\right]\text{cm}^2 = 2\ \text{cm}^2$

原圖的面積 $= 2\,\text{cm}^2 \times 12 = 24\,\text{cm}^2$

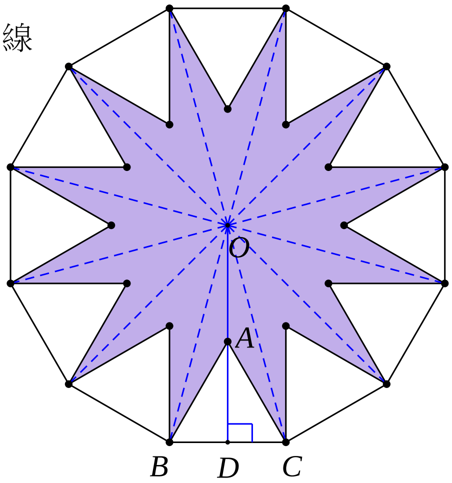


圖 4.2

題目七

在 ΔABC 中，$AB = AC$，$\angle BAC = 20°$，D 是 AB 上一點，使 $AD = BC$ 。求 $\angle BDC$。

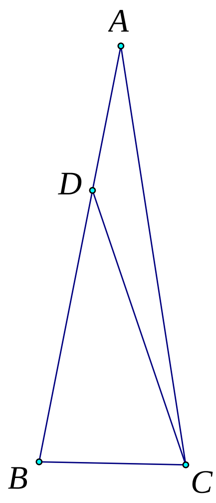


圖 5

方法一

在 B 點的同一側加入一點 X，使 ΔXAC 成為一個等邊三角形。

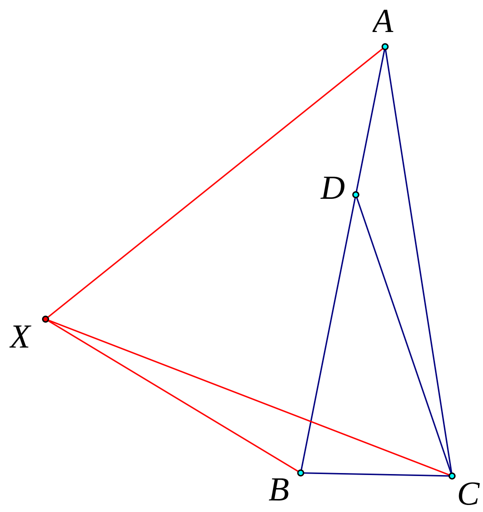


圖 5.1

$\because$ $\angle ACX = 60°$ 及 $\angle ACB = 80°$

$\therefore$ $\angle XCB = 20°$，即 $\angle CAD = \angle XCB$。

已知 $AD = CB$ 和 $AC = CX$，

所以 $\Delta ADC \cong \Delta CBX$ （S.A.S.）。

由此得 $\angle ACD = \angle CXB$。

另一方面，$AX = AB = AC$，即 X、B、C 三點位於一個以 A 點為圓心，AC 為半徑的圓之上，而 $\angle CXB$ 則為該圓上一圓周角。

因為圓心角 $\angle BAC = 20°$，所以
$\angle ACD = \angle BXC = 20° \div 2 = 10°$。

最後，由 ΔADC 的外角關係可得
$\angle BDC = 20° + 10° = 30°$。

方法二

在直線 AC 線上加一點 E，使 $AE = AD\ (= BC)$。

以 AC 為對稱軸，將 D 點反射至 D'。
又以 AB 為對稱軸，將 E 點反射至 E'。

由此 $AE' = AE = AD = AD'$
及 $\angle E'AD' = \angle E'AD + \angle DAE + \angle EAD' = 60°$，

所以 $\Delta AE'D'$ 為等邊三角形。故此，$E'D' = BC$。

另外，$\Delta AD'C \cong \Delta ADC \cong \Delta AEB \cong \Delta AE'B$（S.A.S.）。
故此，$E'B = D'C$，即 $E'D'CB$ 為一平行四邊形。

同時，$\angle ACD' = \angle ACD = \angle ABE = \angle ABE'$。

因為 $BE' // CD'$，所以
$(\angle ACD' + 80°) + (80° + \angle ABE') = 180°$，
由此得 $\angle ACD = 10°$。

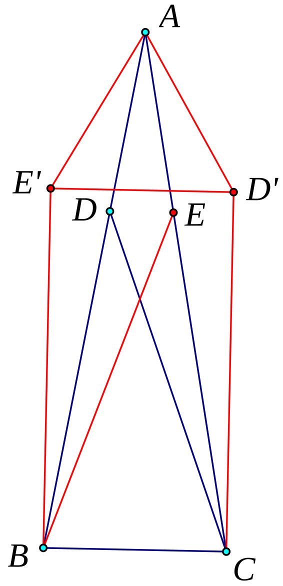


圖 5.2

最後，由 ΔADC 的外角關係可得 $\angle BDC = 20° + 10° = 30°$

方法三

在 ΔABC 內定一點 P 使 ΔPBC 為一等邊三角形。又在 AC 上定一點 Q 使 $QC = BC$。
由此得 $AD = BC = PB = PC = QC$。亦得 $AQ = BD$。

連結 DP 和 DQ。

$\angle PBD = \angle ABC - \angle PBC = 20° = \angle BAC$。
由此可知 $\Delta DAQ \cong \Delta PBD$（S.A.S.），
並且 $DQ = DP$。

再由此推得 $\Delta QDC \cong \Delta PDC$（S.S.S.），
即 $\angle QCD = \angle PCD$。但因為 $\angle PCQ = 20°$，
所以 $\angle QCD = 20° \div 2 = 10°$。

最後，由 ΔADC 的外角關係可得
$\angle BDC = 20° + 10° = 30°$。

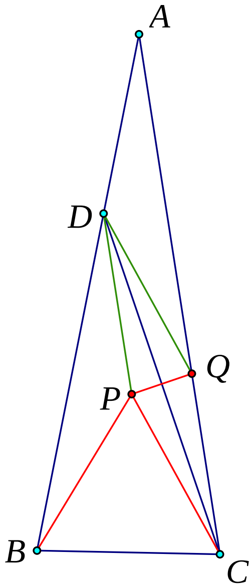


圖 5.3

方法四

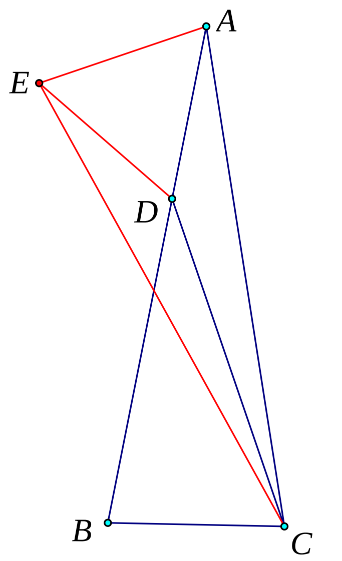


圖 5.4

在 C 點的另一側加入一點 E,使 ΔADE 成為一個等邊三角形。$\angle EAC = \angle EAD + \angle BAC = 80°$，即 $\angle EAC = \angle ACB$。又 $AE = AD = BC$，AC 為公共邊，所以 $\Delta ABC \cong \Delta CEA$（SAS）。由此得 ΔCEA 為等腰三角形，$\angle ACE = 20°$。

再由此得 $\Delta ADC \cong \Delta EDC$（S.S.S.），即 $\angle ACD = \angle ECD$。因為 $\angle ACE = 20°$，

所以 $\angle ACD = 20° \div 2 = 10°$。

最後，由 ΔADC 的外角關係可得 $\angle BDC = 20° + 10° = 30°$。

方法五

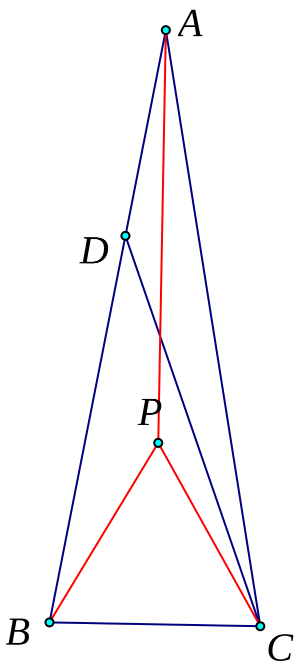


圖 5.5

在 ΔABC 內定一點 P 使 ΔPBC 為一等邊三角形。

連 AP。

$\angle PBA = \angle ABC - \angle PBC = 20°$
$\angle PCA = \angle ACB - \angle PCB = 20°$

$\therefore \angle PBA = \angle PCA = \angle BAC$
$PB = PC = BC = AD$ 和 $AB = AC$

由此可知 $\Delta APB \cong \Delta APC \cong \Delta CDA$（S.A.S.）。
即 $\angle APB = \angle APC = \angle ADC = 150°$。

最後，可得 $\angle BDC = 30°$。

題目八

在 ΔABC 中，$AB = AC$，$\angle BAC = 20°$，M 和 N 分別是 AB 和 AC 上的點，使 $\angle MCB = 60°$，$\angle NBC = 50°$。求 $\angle CMN$。

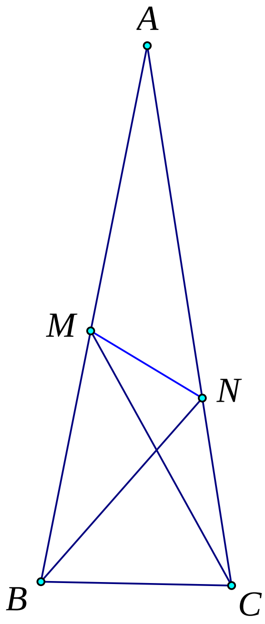


圖 6

方法一

在 MC 上定一點 O，使 ΔOBC 為一等邊三角形。
延長 BO 並交 AC 於 L。連結 LM 及 ON。

$\angle LBN = \angle OBC - \angle NBC = 60° - 50° = 10°$，

$$\angle BNC = 180° - \angle NBC - \angle BCA$$
$$= 180° - 50° - 80° = 50°。$$

$\therefore OC = BC = NC$，即 ΔCNO 為等腰三角形。由此得
$\angle CON = (180° - \angle OCA) \div 2 = 80°$。

又因為 $\angle OBM = \angle LCO = 20°$，$\angle BOM = \angle LOC$ 及 $BO = OC$，所以 $\Delta MOB \cong \Delta LOC$（A.S.A.）。

由此得 $OM = OL$，且 $\angle MOL = \angle BOC = 60°$，可以得知 ΔOLM 為等邊三角形，所以

$$\angle LON = 180° - \angle MOL - \angle CON$$
$$= 180° - 60° - 80° = 40°。$$

從 ΔABL 的外角關係可知 $\angle OLN = 40°$。
綜合上述兩個結果可知 ΔNOL 為等腰三角形。
由此可知 MN 平分 $\angle OML$（$= 60°$），
即 $\angle CMN = 30°$。

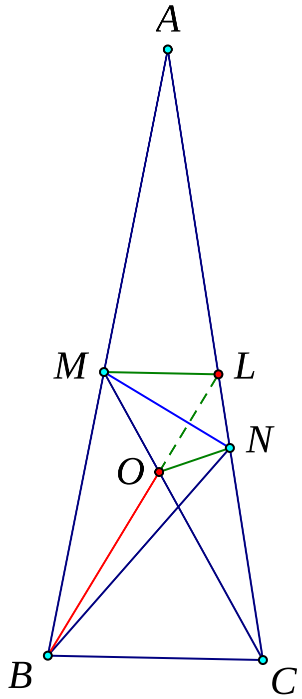


圖 6.1

方法二

在 AB 上取一點 P，使 $\angle BCP = 20°$。

$\because \angle BPC = 180° - \angle BCP - \angle CBP = 80° = \angle CBP$，

$\therefore \Delta CBP$ 為等腰三角形。

$\angle BNC = 50°$，由此得 $PC = BC = NC$。又因為 $\angle PCN = 60°$，所以 ΔPNC 為等邊三角形。

再由此得 $\angle MPN = 180° - \angle NPC - \angle BPC = 40°$。

另外，$\angle PCM = \angle ACB - \angle BCP - \angle MCN = 40°$。

由 ΔPCM 的內角和亦知 $\angle PMC = 40°$，

因此 $PM = PC = PN$。換句話說，ΔPMN 亦為等腰三角形，$\angle PMN = 70°$。

最後，$\angle CMN = 70° - 40° = 30°$。

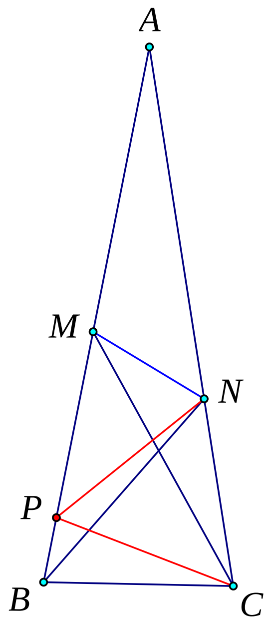


圖 6.2

方法三

由於 $AB = AC$，$\angle ACB = \angle ABC = 80°$。

設 $\angle CMN$ 為 x，則$\angle CNM = 160° - x$。

在 ΔCNM，依據正弦公式 (Sine formula)，

$CM : CN = \sin(160° - x) : \sin x$

在 ΔBCM，$CM : BC = \sin 80° : \sin 40° = 2\cos 40° \sin 40° : \sin 40° = 2\cos 40°$

由於$\angle BNC = \angle NBC = 50°$，$\Delta BCN$ 為等腰三角形，

$CN = BC \Rightarrow \sin(160° - x) : \sin x = \sin 80° : \sin 40°$

$$\sin(180° - (20° + x)) : \sin x = 2\cos 40°$$

$$\sin(20° + x) = 2\cos 40° \cdot \sin x$$

$$\sin(20° + x) = 2\cos(60° - 20°) \cdot \sin x$$

$$\sin 20° \cdot \cos x + \cos 20° \cdot \sin x = 2\left(\frac{1}{2}\cos 20° + \frac{\sqrt{3}}{2}\sin 20°\right)\sin x$$

$$\cot x = \sqrt{3}$$

$$x = 30°$$

此道問題的其他題解，同學可參考以下網址：

http://www.cut-the-knot.org/triangle/80-80-20/IndexToClassical.shtml

題目九

證明對任意實數 x 不等式 $x^6 - x^3 + x^2 - x + 1 > 0$ 均成立。

方法一

$$x^6 - x^3 + x^2 - x + 1 = \left(x^3 - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$$

當 x 為任意實數，$\left(x^3 - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^2 + \frac{1}{2} > 0$

∴ 不等式成立。

方法二：分情況討論

(1) 當 $x \geq 1$， $x^6 \geq x^3$ ， $x^2 \geq x$

∴ $x^6 - x^3 + x^2 - x + 1 > 0$ … (1)

(2) 當 $0 \leq x < 1$ ，$x^3 \leq x^2$ ，$x^6 - x^3 + x^2 - x + 1 = x^6 + (-x^3 + x^2) + (-x + 1) > 0$

(3) 當 $x < 0$，(1) 的左邊各項均為正數，正數必為大於 0，所以(1)式成立。

題目十

A、B、C、D、E 五個人站成一排，如果 B 必須站在 A 的右方，那麼共有多少種不同的排法？

方法一

以 O 表示 B 可站的位置，A 的站法可分四類，如下表所示：

	位置標號					B 可選一個"O"站立的位置，相應站法	C、D、E 站餘下位置的相應站法	總站法數目
	1	2	3	4	5			
Ⅰ	A	O	O	O	O	C_1^4	P_3^3	$C_1^4 \cdot P_3^3$
Ⅱ	※	A	O	O	O	C_1^3	P_3^3	$C_1^3 \cdot P_3^3$
Ⅲ	※	※	A	O	O	C_1^2	P_3^3	$C_1^2 \cdot P_3^3$
Ⅳ	※	※	※	A	O	1	P_3^3	$1 \cdot P_3^3$
							共	60

方法二

關注"A、B 可以不相鄰"，就"A、B 相鄰與否"分成兩類解決。

(i) A、B 相鄰

先把 A、B 合併為一元素，讓其與 C、D、E 的全排列，一共有 P_4^4 種排法，然後，我們再考慮 A、B 的順序，但是 B 必須站在 A 的右邊，於是，只有 1 種站法，由此得知： A、B 相鄰的站法共有 P_4^4 種。

(ii) A、B 不相鄰

第一步，排 C、D、E，有 P_3^3 種方法，

第二步 在 四個空位置中選兩個空位置讓 A、B 站立，共有 C_2^4 種方法，由此得知： A、B 不相鄰的站法共有 $C_2^4 \cdot P_3^3$ 種。

所以共 $P_4^4 + C_2^4 \cdot P_3^3 = 60$ 種方法。

方法三

在沒有任何條件下，把 A、B、C、D、E 五人安排在五個不同的位置上，排法共有 P_5^5 種。而 A、B 的排列方法共 P_2^2 種.，故 "B 必須站在 A 的右邊"的站法共有 $\frac{P_5^5}{P_2^2} = 60$ 種

方法四

在五個不同位置中選兩個給 A、B 有 C_2^5 種選法，又 A、B 有定序，則每一種選法對應着 A、B 的一種站法，故 A、B 的站法數目也即是 C_2^5。在餘下的不同位置上排 C、D、E，有 P_3^3 種排法。故共有 $C_2^5 \cdot P_3^3 = 60$ 種。

Question 11

If h and k are real numbers such that $h^2 - 7hk + k^2 = 2$, show that $h^2 + k^2 \geq \frac{4}{9}$.

Method 1

Let $W = h^2 + k^2$.

By arithmetic mean ≥ geometric mean*, we get

$$\frac{h^2 + k^2}{2} \geq \sqrt{h^2 k^2}$$

$$2|hk| \leq h^2 + k^2$$

$$-(h^2 + k^2) \leq 2hk \leq h^2 + k^2 \quad \text{i.e.} \quad -W \leq 2hk \leq W \ \ldots (1)$$

On the other hand, $h^2 - 7hk + k^2 = 2$

$$hk = \frac{W-2}{7} \qquad \ldots (2)$$

Substituting (2) into (1), $-W \leq \frac{2(W-2)}{7} \leq W$

which gives $W \geq \frac{4}{9}$

Method 2

Let $W = h^2 + k^2$.

$h^2 - 7hk + k^2 = 2$ gives $hk = \frac{W-2}{7}$.

$$0 \leq (h-k)^2 = h^2 - 2hk + k^2 = W - \frac{2(W-2)}{7} = \frac{5W+4}{7}$$

AND

$$0 \leq (h+k)^2 = h^2 + 2hk + k^2 = W + \frac{2(W-2)}{7} = \frac{9W-4}{7}$$

$\therefore \quad W \geq \frac{4}{9}$

Remark: The proof of * is given in Question 25.

Method 3

Since $h^2 + k^2 = 2 + 7\,hk$, we are going to find the range of values of hk.

Let $x = h - k$.

$x^2 = h^2 - 2hk + k^2 = h^2 - 7hk + k^2 + 5hk = 2 + 5hk$

Then $hk = \dfrac{x^2 - 2}{5}$ and $h = x + k$.

Substituting into $h^2 - 7hk + k^2 = 2$, we get

$$(x + k)^2 - 7\left(\frac{x^2 - 2}{5}\right) + k^2 = 2$$

$$5k^2 + 5xk + 2 - x^2 = 0$$

$\because$ It is a quadratic equation with real roots k.

$\therefore$ Discriminant ≥ 0

$$(5x)^2 - 4\,(5)(2 - x^2) \geq 0$$

$$x^2 \geq \frac{8}{9}$$

$h^2 + k^2 = 2 + 7\,hk$

$$= 2 + 7\left(\frac{x^2 - 2}{5}\right)$$

$$= \frac{7x^2 - 4}{5}$$

$$\geq \frac{7\left(\frac{8}{9}\right) - 4}{5}$$

$$= \frac{4}{9}$$

Method 4

First note that (h, k) is a variable point on the conic $h^2 - 7hk + k^2 = 2$.
Rotate the conic in an anti-clockwise direction about the origin through an angle of $\frac{\pi}{4}$. Let (u, v) be the image of (h, k) after the rotation.

Then $$\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(u+v) \\ \frac{1}{\sqrt{2}}(-u+v) \end{pmatrix}.$$

Substituting into $h^2 - 7hk + k^2 = 2$, we get

$$\left[\frac{1}{\sqrt{2}}(u+v)\right]^2 - 7 \times \frac{1}{\sqrt{2}}(u+v) \times \frac{1}{\sqrt{2}}(-u+v) + \left[\frac{1}{\sqrt{2}}(-u+v)\right]^2 = 2$$

$$9u^2 - 5v^2 = 4$$

As $h^2 + k^2$ is the square of the distance of (h, k) from the origin and this distance remains unchanged during the rotation, the minimum value of $h^2 + k^2$ is equal to the square of the distance of its new vertex $\left(\frac{2}{3}, 0\right)$ from the origin, which is given by $\frac{4}{9}$, i.e. $h^2 + k^2 \geq \frac{4}{9}$.

Question 12

Given that $P(x, y)$ is a point on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. Let $k = 2x - y$. Find the maximum value of k.

Method 1

Substituting $y = 2x - k$ into $\frac{x^2}{4}+\frac{y^2}{9}=1$, we get

$$9x^2 + 4(2x - k)^2 = 36$$

$$25x^2 - 16kx + 4k^2 - 36 = 0$$

As it is a quadratic equation with real roots x, discriminant ≥ 0

$$(-16k)^2 - 4(25)(4k^2 - 36) \geq 0$$

$$0 \leq k^2 \leq 25$$

$\therefore$ The maximum value of k is 5.

Method 2

As $P(x, y)$ is a point on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, let $x = 2\cos\theta$ and $y = 3\sin\theta$.

Then $k = 2x - y = 2(2\cos\theta) - 3\sin\theta$

$= 4\cos\theta - 3\sin\theta$

$= 5\cos(\theta + \alpha)$ where $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

$\because\ -1 \leq \cos(\theta + \alpha) \leq 1$

$\therefore$ The maximum value of k is 5.

Method 3

Let $\vec{a} = \frac{x}{2}\vec{i} + \frac{y}{3}\vec{j}$ and $\vec{b} = 4\vec{i} - 3\vec{j}$.

$\because\ \vec{a}\bullet\vec{b} \leq |\vec{a}||\vec{b}| \quad \therefore\ |\vec{a}|^2 \geq \frac{\left(\vec{a}\bullet\vec{b}\right)^2}{|\vec{b}|^2}$

$$\frac{x^2}{4}+\frac{y^2}{9} \geq \frac{(2x-y)^2}{4^2+(-3)^2}$$

$$1 \geq \frac{(2x-y)^2}{25}$$

$$\therefore\quad 2x - y \leq 5$$

i.e. The maximum value of k is 5.

Question 13

Find the range of values of y if $y = \dfrac{2-\sin x}{2-\cos x}$.

Method 1

$$y = \frac{2-\sin x}{2-\cos x}$$

$\sin x - y\cos x = 2 - 2y$

$\sqrt{1+y^2}\sin\left(x-\alpha\right) = 2-2y$ where $\cos\alpha = \dfrac{1}{\sqrt{1+y^2}}$ and $\sin\alpha = \dfrac{y}{\sqrt{1+y^2}}$.

$$\sin\left(x-\alpha\right) = \frac{2-2y}{\sqrt{y^2+1}}$$

$$\therefore \quad -1 \le \frac{2-2y}{\sqrt{y^2+1}} \le 1$$

$$3y^2 - 8y + 3 \le 0$$

$$\frac{4-\sqrt{7}}{3} \le y \le \frac{4+\sqrt{7}}{3}$$

Method 2

$P(\cos x, \sin x)$ is a movable point on the circle $u^2 + v^2 = 1$ and $A(2,2)$ is a fixed point. Then let k = slope of AP and the equation of the line passing though A with slope k is $v - 2 = k\left(u-2\right)$ i.e. $ku - v - 2k + 2 = 0$.

The distance* from the centre of the circle (0,0) to this line is less than or equal to the radius.

$$\therefore \quad \frac{\left|-2k+2\right|}{\sqrt{k^2+(-1)^2}} \le 1$$

$$\left(-2k+2\right)^2 \le k^2 + 1$$

$$\frac{4-\sqrt{7}}{3} \le k \le \frac{4+\sqrt{7}}{3}$$

Remark: * The formula for finding the distance from a point to a line is shown in Question 24.

Question 14

In ΔABC, $AB = c$, $BC = a$ and $CA = b$ such that $c^2 = a^2 + ab$.

Prove that $\angle BCA = 2\angle BAC$.

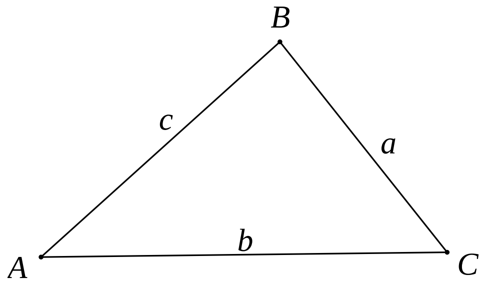


Figure 7

Method 1

Extend BC to D such that $CD = AC$.

From $c^2 = a^2 + ab$, $\dfrac{c}{a} = \dfrac{a+b}{c}$ and $\angle ABC = \angle ABD$ (common)

$\therefore$ $\Delta ABC \sim \Delta DBA$ (2 sides proportional, inc. $\angle$)

$\therefore$ $\angle BAC = \angle BDA$ (corr $\angle$s, ~Δs)

$\because$ $CA = CD = b$

$\therefore$ $\angle CAD = \angle CDA$ (base $\angle$s, isos. Δ)

$\therefore$ $\angle BCA = \angle CAD + \angle CDA$ (ext. $\angle$ of Δ)

$= 2\angle CDA$

$= 2\angle BDA$

$= 2\angle BAC$

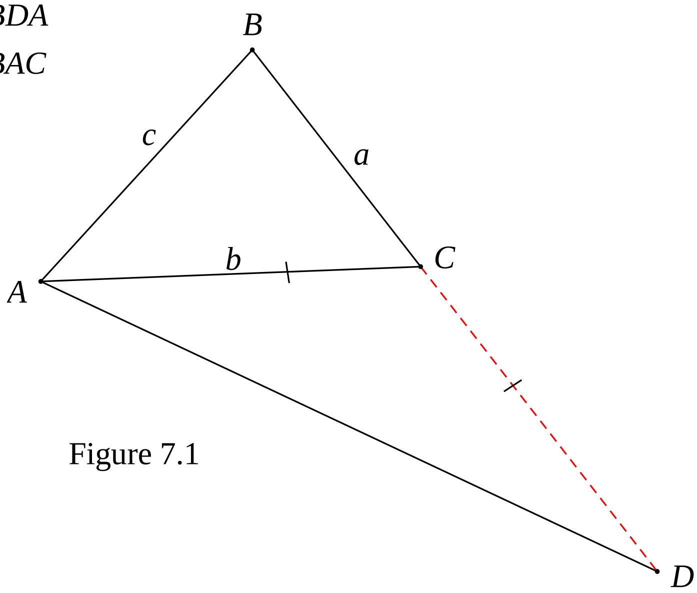


Figure 7.1

Method 2

Let the angle bisector of $\angle BCA$ be CD, and let $\angle BCD = \angle DCA$ be θ and $BD = x$.
Then $DA = c - x$.

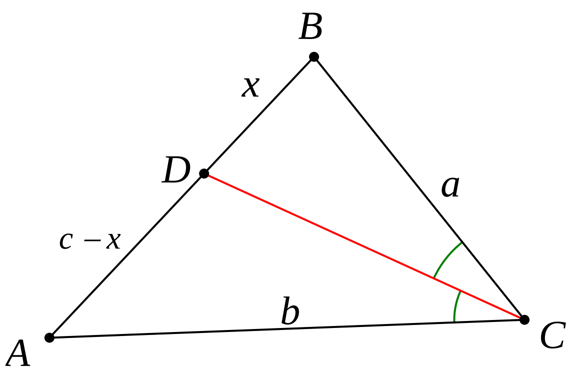


Figure 7.2

$$\frac{\text{Area of } \Delta ACD}{\text{Area of } \Delta CBD}$$

$$= \frac{\frac{1}{2} AC \times CD \sin\theta}{\frac{1}{2} BC \times CD \sin\theta} = \frac{c-x}{x}$$

$$\frac{b}{a} = \frac{c-x}{x}$$

$$x = \frac{ac}{a+b}$$

$$\therefore \quad BD \times AB = x \cdot c = \frac{ac^2}{a+b} = \frac{a\left(a^2 + ab\right)}{a+b} = a^2 \quad = BC^2$$

i.e. $\frac{BC}{AB} = \frac{BD}{BC}$

Also $\angle ABC = \angle CBD$

$\therefore \quad \Delta ABC \sim \Delta CBD$ (2 sides proportional, inc. $\angle$)

Then $\angle BCA = \angle BDC$ (corr. $\angle$s, ~Δs)

$2\theta = \angle BAC + \angle DCA$ (ext. $\angle$ of Δ)

$2\theta = \angle BAC + \theta$

$\angle BAC = \theta$

$\therefore \quad \angle BCA = 2\ \angle BAC$.

Question 15

Figure 8 shows a square *ABCD*. *E* is the mid-point of *CD*, *F* is a point which lies on *BC*. Also $AF = CD + CF$. Prove *AE* is the angle bisector of $\angle DAF$.

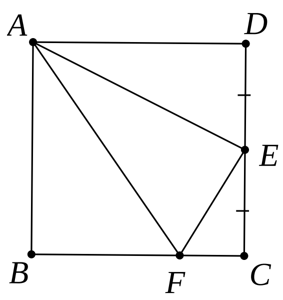


Figure 8

Method 1

Produce *FE* to *G* such that *ADG* is a straight line.

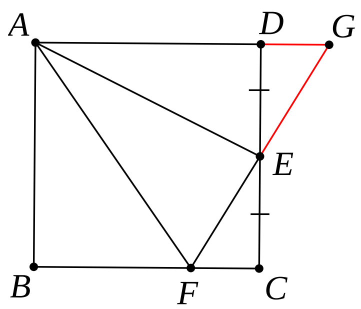


Figure 8.1

In ΔDEG and ΔCEF,

$DE = EC$ (given)

$\angle DEG = \angle FEC$ (vert opp $\angle$)

$\angle EDG = 180° - \angle ADE$ (adj. $\angle$s on a st. line)

$= 90°$ (definition of square)

$= \angle ECF$ (definition of square)

$\therefore$ $\Delta DEG \cong \Delta CEF$ (A.S.A.)

$\therefore$ $DG = CF$ (corr. sides, $\cong\Delta$s)

$AG = AD + DG$

$= CD + CF$ (definition of square)

$= AF$ (given)

$\because$ $EF = EG$ (corr. sides, $\cong\Delta$s)

$\therefore$ *AE* is the median of an isosceles ΔAFG.

i.e. *AE* is the angle bisector of $\angle DAF$.

Method 2

Produce *AE* and *BC* such that they meet at *N*.

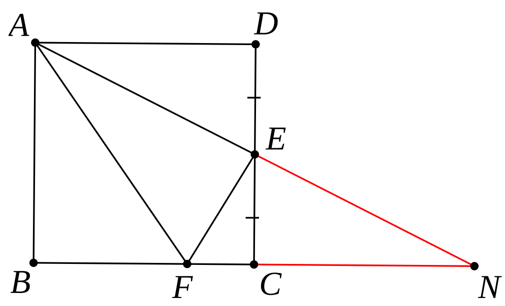


Figure 8.2

It is easy to show that $\Delta ADE \cong \Delta NCE$ (A.S.A.).

$\therefore$ $AD = NC$ (corr. sides, $\cong\Delta$s)

and $AF = CD + CF$ (given)

$= AD + CF$ (definition of square)

$= NC + CF$

$= FN$

$\therefore \angle FAN = \angle FNA$ (base $\angle$s, isos. Δs)

$\angle FNA = \angle DAE$ (alt. $\angle$s, $AD // CN$)

$\therefore \angle FAE = \angle DAE$ i.e. *AE* is the angle bisector of $\angle DAF$.

Question 16

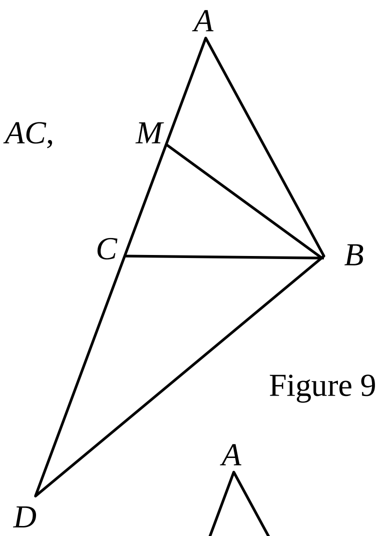


Figure 9

In ΔABC, $AC = AB$. If $AC = CD$ and M is the mid-point of AC, is it true that $BD = 2BM$?

Method 1

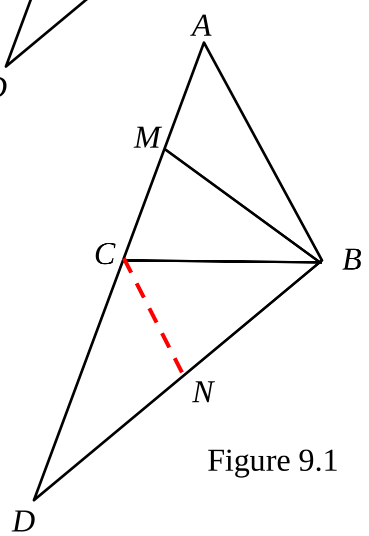


Figure 9.1

Let N be the mid-point of BD.

$\because$ $AC = CD$ and $BN = ND$,

$\therefore$ $CN = \frac{1}{2}AB$ and $CN \,//\, AB$ (Mid-point theorem)

$\therefore$ $\angle NCB = \angle ABC$ (alt. $\angle$s, $CN \,//\, AB$)

$\angle ABC = \angle ACB$ (base $\angle$s, isos. Δ)

Then $\angle NCB = \angle MCB$

$\because$ M is the mid-point of AC

$\therefore$ $CM = \frac{1}{2}AC = \frac{1}{2}AB = CN$

Also BC is the common side.

$\therefore$ $\Delta CNB \cong \Delta CMB$ (S.A.S.)

$\therefore$ $BN = BM$ (corr. sides, $\cong \Delta$s)

i.e. $BM = BN = \frac{1}{2}BD$

Method 2

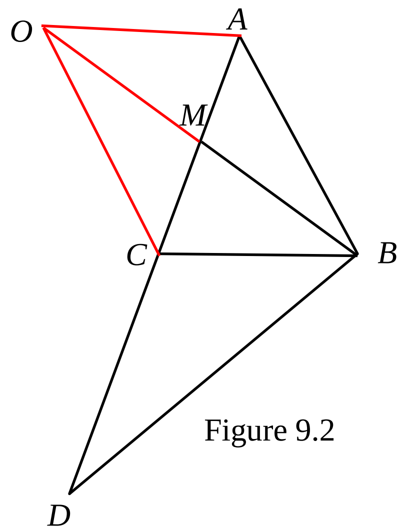


Figure 9.2

Extend BM to O so that $OM = BM$. Join OA and OC.

Then $OABC$ is a parallelogram. (diagonals bisect each other)

$\therefore OA = BC$ (opp. sides, //gram)

$AB = AC = CD$ (given)

$\angle BCD = 180° - \angle ACB$ (adj. $\angle$ on a st. line)

$\angle BAO = 180° - \angle ABC$ (int. $\angle$s, OA // BC)

$\because \angle ACB = \angle ABC$ (base $\angle$s, isos. Δ)

$\therefore \angle BCD = \angle BAO$

$\therefore$ $\Delta BCD \cong \Delta OAB$ (S.A.S.)

Then $BD = OB$ (corr. sides, $\cong \Delta$s)

$\therefore$ $BD = 2BM$

Question 17

In Figure 10, $BD \perp CD$, $BF // CD$ and $EF = 2BC$. Is it true that $\angle BCD = 3\angle FCD$?

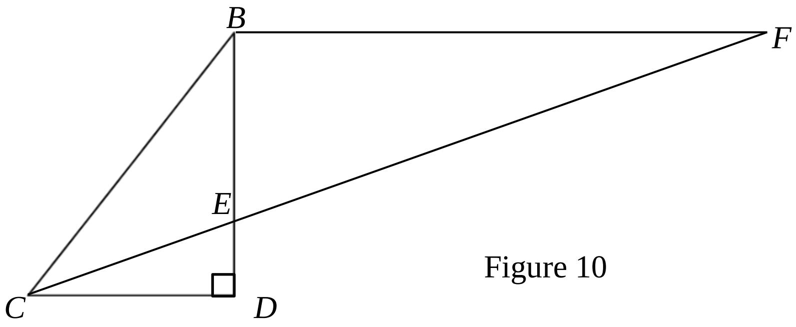


Figure 10

Method 1

Let O be the mid-point of EF. Join BO.

$\because$ $\angle FBD = \angle BDC = 90°$ (alt. $\angle$s, $BF // CD$)

$\therefore$ We can construct a circle passing through B, E and F with EF as the diameter and O as the centre.

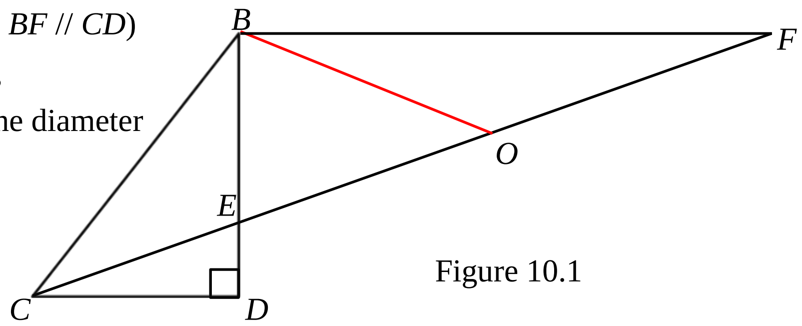


Figure 10.1

Then $BO = \frac{1}{2}EF = BC$

$\therefore BO = OF = BC$

$\therefore$ $\angle BFO = \angle FBO$ and $\angle BCO = \angle BOC$ (base $\angle$s, isos.Δ)

$\angle BOC = \angle BFO + \angle FBO = 2\angle BFO$ (ext. $\angle$ of Δ)

$\therefore$ $\angle BCO = 2\angle BFO$

Also $\angle ECD = \angle BFO$ (alt. $\angle$s, $BF//CD$)

$\therefore$ $\angle BCD = \angle BCO + \angle ECD = 3\angle FCD$

Method 2

Let $BG \perp CF$, $\angle FCD = \alpha$, $\angle BCE = \beta$, $BC = a$.

Then in ΔBCG, $BG = a\sin\beta$

In ΔBEF,

$\angle FBD = \angle BDC = 90°$ and

$\angle BFE = \angle FCD = \alpha$ (alt.$\angle$s, BF//CD)

$BF = 2a\cos\alpha$

In ΔBFG, $BG = 2a\cos\alpha\sin\alpha$

$\therefore$ $a\sin\beta = 2a\cos\alpha\sin\alpha$

$\therefore$ $\sin\beta = \sin 2\alpha$.

$\beta = 2\alpha$

i.e. $\angle BCD = 3\angle FCD$

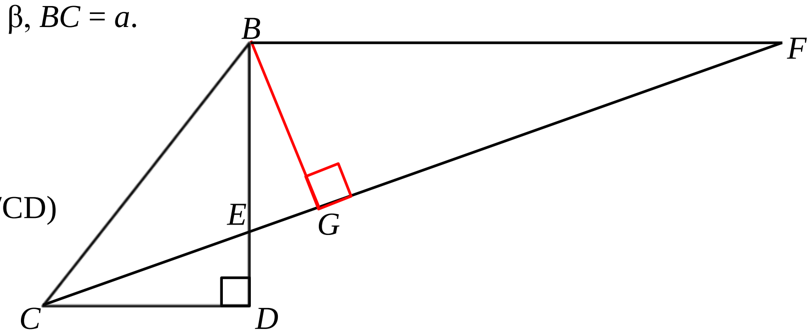


Figure 10.2

Question 18

Given that *ABCD* is a square, and *O* is a point inside the square such that $\angle ODC = \angle OCD = 15°$. Prove that *AOB* is an equilateral triangle.

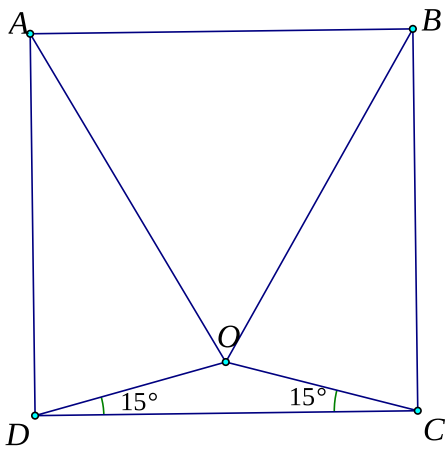


Figure 11

Method 1

Construct $\Delta PAD \cong \Delta ODC$ inside *ABCD*.

$DO = CO$ (sides opp. eq. ∠s)

$\therefore AP = PD = DO = OC$

Also $\angle PAD = \angle PDA = \angle ODC = 15°$

$\angle ADC = 90°$ (definition of a square)

$\therefore \angle PDO = 60°$

$\angle DPO = \angle DOP$ (base ∠s of isos. Δ)

$\therefore \angle DPO = 60°$ (∠ sum of Δ)

$\angle APD = 150°$ (∠ sum of Δ)

$\therefore \angle APO = 150°$ (∠s at a point)

$\angle PAO = \angle POA$ (base ∠s, isos. Δ)

$\therefore \angle PAO = 15°$ (∠ sum of Δ)

$\angle DAB = 90°$ (definition of a square)

$\therefore \angle OAB = 60°$

Similarly, $\angle OBA = 60°$

$\therefore \angle AOB = 60°$ (∠ sum of Δ)

$\therefore \Delta AOB$ is equilateral.

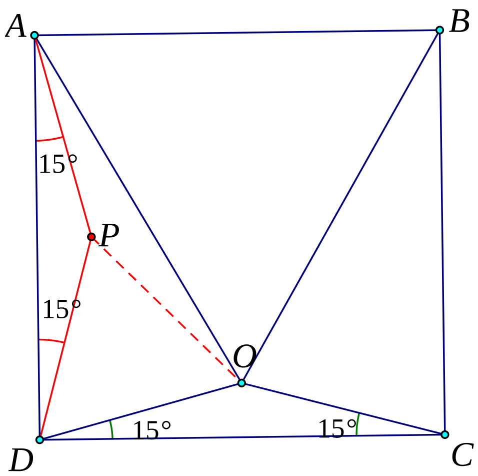


Figure 11.1

Method 2

$\angle ADC = \angle BCD = 90°$ (definition of square)

$\angle ADO = \angle BCO = 75°$

$DO = CO$ (sides opp. eq. $\angle$s)

$AD = BC$ (definition of square)

$\therefore \Delta ADO \cong \Delta BCO$ (S.A.S.)

$\therefore$ $AO = BO$ (corr. sides, $\cong \Delta$s)

$\angle AOD = \angle BOC$ (corr. $\angle$s, $\cong \Delta$s)

Let $\angle OAB = x$.

Then $\angle OBA = x$ (base $\angle$s of isos. Δ)

$\angle AOB = 180° - 2x$ ($\angle$ sum of Δ)

$\angle DOC = 150°$ ($\angle$ sum of Δ)

$\therefore$ $\angle AOD = x + 15°$ and $\angle ADO = 75°$

If $x < 60°$ then $\angle AOD < \angle ADO = 75°$. $\therefore$ $AD < AO$.

Also $\angle ABO < \angle AOB$ $\therefore$ $AO < AB$.

i.e. $AD < AB$ which contradicts the fact that $ABCD$ is a square.

$\therefore$ x cannot be smaller than 60°.

Similarly, x cannot be larger than 60°.

$\therefore$ $x = 60°$

$\therefore$ $\angle ABO = \angle BAO = \angle AOB = 60°$

$\therefore$ ΔAOB is equilateral.

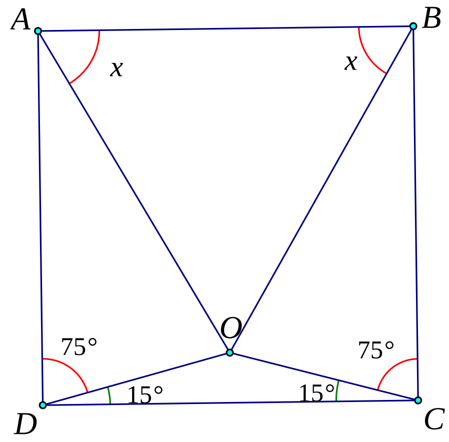


Figure 11.2

Method 3

Produce CO to meet BD at P. Join AP.

$\angle ADP = \angle CDP = 45°$ (property of square)

$DP = DP$ (common)

$AD = DC$ (definition of square)

$\therefore$ $\Delta ADP \cong \Delta CDP$ (S.A.S.)

$\therefore$ $AP = CP$ (corr. sides, $\cong \Delta$s)

$\angle DAP = \angle DCP = 15°$ (corr. $\angle$s, $\cong \Delta$s)

$\angle DPA = \angle DPC$ (corr. $\angle$s, $\cong \Delta$s)

$\angle PDO = 30°$

$\angle POD = 30°$ (ext. $\angle$ of Δ)

$\therefore$ $PO = PD$ (sides opp. eq. $\angle$s)

$\angle DPO = 120°$ ($\angle$ sum of Δ)

$\angle APO = 120°$ ($\angle$s at a point)

$\therefore$ $\Delta APO \cong \Delta CPD$ (S.A.S.)

$\therefore$ $AO = CD$ (corr. sides, $\cong \Delta$s)

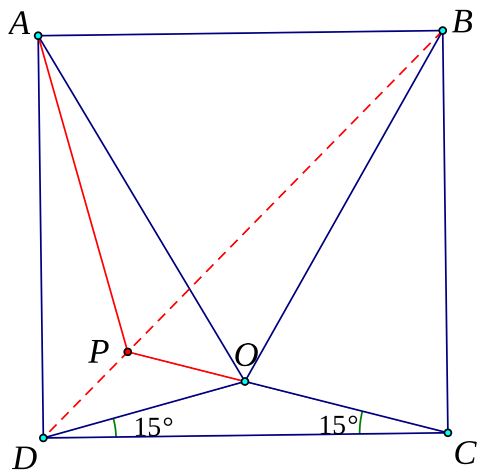


Figure 11.3

Similarly $BO = CD$

$\therefore$ $AO = BO = CD = AB$

$\therefore$ ΔAOB is equilateral.

Method 4

Construct an equilateral ΔCDP as shown.

$DO = CO$	(sides opp. eq. $\angle$s)
$OP = OP$	(common sides)
$DP = CP$	(sides of equil. Δ)
$\therefore \quad \Delta DOP \cong \Delta COP$	(S.S.S.)
$\angle DPO = \angle CPO = 30°$	(corr. $\angle$s, $\cong \Delta$s)
$\angle ADO = 75° = \angle PDO$	
$AD = PD = CD$	(definition of square)
$DO = DO$	(common)
$\therefore \quad \Delta ADO \cong \Delta PDO$	(S.A.S.)
$\therefore \quad \angle DAO = \angle DPO = 30°$	(corr. $\angle$s, $\cong \Delta$s)
$\therefore \quad \angle BAO = 60°$	
Similarly $\angle ABO = 60°$	
$\therefore \quad \angle AOB = 60°$	($\angle$ sum of Δ)
$\therefore \quad \Delta AOB$ is equilateral.	

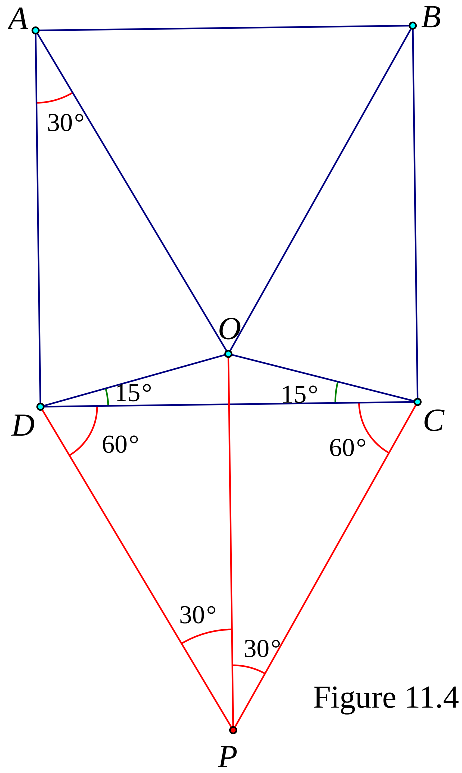


Figure 11.4

Method 5

Construct an equilateral ΔCPD as shown.

$\angle PCD = 60°$	($\angle$ of an equil. Δ)
$\therefore \quad \angle BCP = 30°$	(definition of square)
$\because \quad PC = CD = BC$	
$\therefore \angle BPC = \angle PBC = 75°$	(base $\angle$s of isos. Δ)
$\angle PBA = 90° - 75° = 15°$	(definition of square)
Similarly, $\angle BAP = 15°$	
$AB = CD$	(definition of square)
$\therefore \quad \Delta ABP \cong \Delta DOC$ (ASA)	
$\therefore \quad PB = OC$	(corr. sides, $\cong \Delta$s)
$PO = PO$	(common)

It is easy to show that $\Delta PDO \cong \Delta PCO$ (S.S.S.)

$\therefore \quad \angle CPO = \angle DPO = 30°$	(corr. $\angle$s, $\cong \Delta$s)
$\angle BPO = \angle COP = 105°$	
$\therefore \quad \Delta PBO \cong \Delta OCP$	(S.A.S.)
$\therefore \quad OB = CP$	(corr. sides, $\cong \Delta$s)
$OB = CP = CB = CD$	

Similarly $OA = DP = DA = AB$

$\therefore \quad \Delta AOB$ is equilateral.

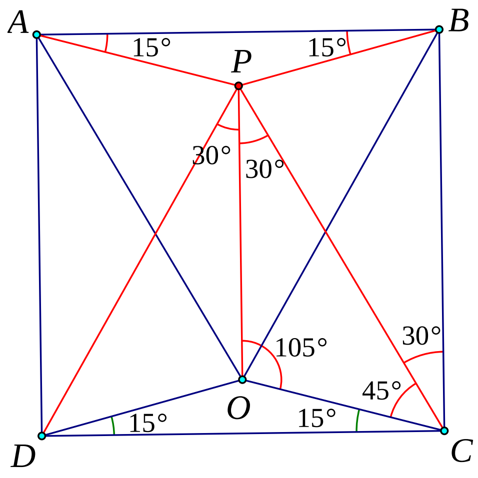


Figure 11.5

Method 6

Apply sine and cosine laws.

Let $AB = BC = CD = DA = 1$. (definition of square)

In ΔOCD, $\dfrac{1}{\sin 150^\circ} = \dfrac{s}{\sin 15^\circ}$

$\therefore\ s = \dfrac{\sin 15^\circ}{\sin 150^\circ} = \dfrac{\sin 15^\circ}{\sin 30^\circ} = \dfrac{1}{2\cos 15^\circ}$

$\angle BCO = 90^\circ - 15^\circ = 75^\circ$ (definition of square)

$$
\begin{aligned}
t^2 &= 1^2 + s^2 - 2\times 1\times s\times \cos 75^\circ \\
&= 1 + \frac{1}{4\cos^2 15^\circ} - 2\times \frac{1}{2\cos 15^\circ}\times \sin 15^\circ \\
&= 1 + \frac{1}{4\cos^2 15^\circ} - 2\times \frac{\sin 15^\circ \cos 15^\circ}{2\cos^2 15^\circ} \\
&= 1 + \frac{1}{4\cos^2 15^\circ} - \frac{2\sin 15^\circ \cos 15^\circ}{2\cos^2 15^\circ} \\
&= 1 + \frac{1}{4\cos^2 15^\circ} - \frac{\sin 30^\circ}{2\cos^2 15^\circ} \\
&= 1
\end{aligned}
$$

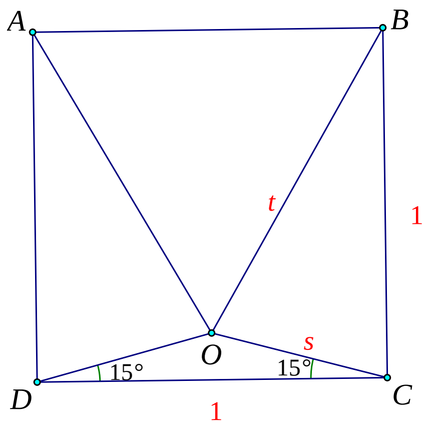


Figure 11.6

$\therefore\ t = 1$

Similarly, $AO = 1$

$\therefore\ OB = OA = AB = 1$

$\therefore\ \Delta AOB$ is equilateral.

Method 7

Construct an equilateral triangle AQB with Q inside the square.

$\angle QAD = 90^\circ - 60 = 30^\circ$ (definition of square)

$\because\ AB = AQ = AD$

$\therefore\ \angle ADQ = \angle AQD = 75^\circ$ (base $\angle$s, isos. Δ)

$\therefore\ \angle QDC = 15^\circ$

Similarly, $\angle QCD = 15^\circ$.

Hence QD, QC coincide OD, OC respectively, i.e. Q and O coincide.

$\therefore\ \Delta AOB$ is same as $\Delta\ AQB$, which is equilateral.

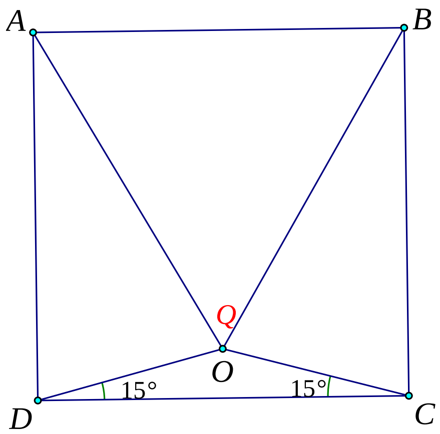


Figure 11.7

Question 19

Prove sin(A + B) = sin A cos B + cos A sin B.

Method 1 $0 < A + B < \frac{\pi}{2}$

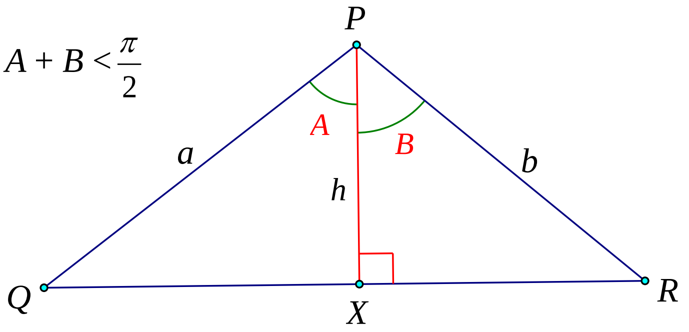


Figure 12

Consider ΔPQR. Construct a line PX such that $PX \perp QR$.

Let $\angle QPX = A$, $\angle XPR = B$, $QP = a$, $PR = b$, $PX = h$

In ΔPQX, $h = a \cos A$ … (1)

In ΔPRX, $h = b \cos B$ … (2)

Consider the areas of ΔPQX, ΔPRX, ΔPQR.

From (2), area of ΔPQX $= \frac{1}{2} ah \sin A = \frac{1}{2} ab \sin A \cos B$

From (1), area of ΔPRX $= \frac{1}{2} bh \sin B = \frac{1}{2} ab \sin B \cos A$

Area of ΔPQR $= \frac{1}{2} ab \sin(A + B)$

$\because$ Area of ΔPQR = Area of ΔPQX + Area of ΔPRX

$$\frac{1}{2} ab \sin(A + B) = \frac{1}{2} ab \sin A \cos B + \frac{1}{2} ab \sin B \cos A$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

Method 2

Let $z_1 = \cos A + i \sin A$ and $z_2 = \cos B + i \sin B$ where $i^2 = -1$.

Then the exponential form of z_1 and z_2 are respectively e^{iA} and e^{iB}.

$\therefore$ $e^{iA} \times e^{iB} = (\cos A + i \sin A)(\cos B + i \sin B)$

$e^{i(A+B)} = (\cos A \cos B - \sin A \sin B) + i(\sin A \cos B + \cos A \sin B)$

On the other hand, $e^{i(A+B)} = \cos(A + B) + i \sin(A + B)$

$\therefore$ $\cos(A + B) = \cos A \cos B - \sin A \sin B$

And $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Method 3

Let $\overrightarrow{OP} = (\cos A)\vec{i} + (\sin A)\vec{j}$ and $\overrightarrow{OQ} = (\cos B)\vec{i} + (\sin B)\vec{j}$

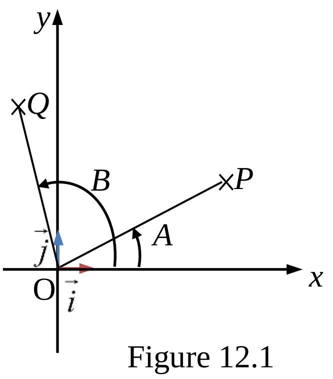


Figure 12.1

$\overrightarrow{OP} \bullet \overrightarrow{OQ} = \cos A \cos B + \sin A \sin B$ and

$\overrightarrow{OP} \bullet \overrightarrow{OQ} = \left|\overrightarrow{OP}\right| \left|\overrightarrow{OQ}\right| \cos \angle POQ = \cos(B - A) = \cos(A - B)$

$\therefore$ cos (A – B) = cos A cos B + sin A sin B

Replace A by $\frac{\pi}{2} - A$.

$$\cos\left(\frac{\pi}{2} - (A + B)\right) = \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Method 4

In Figure 12.1, $P = (\cos A, \sin A)$ and $Q = (\cos B, \sin B)$.

Then $OP = OQ = 1$

By distance formula,

$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$

$= 2 - 2(\cos A \cos B + \sin A \sin B)$

By cosine formula,

$PQ^2 = OP^2 + OQ^2 - 2(OP)(OQ) \cos (A - B)$

$= 2 - 2\cos (A - B)$

$\therefore$ cos (A – B) = cos A cos B + sin A sin B

Replace A by $\frac{\pi}{2} - A$.

$$\cos\left(\frac{\pi}{2} - (A + B)\right) = \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Question 20

In ΔABC, $BC = a$, $AB = c$, $AC = b$ and $\angle ABC = 2\angle BAC$. Prove that $b^2 = a(a + c)$.

Method 1

Construct the angle bisector BD of $\angle ABC$. Let $\angle BAC = x$.

Then $\angle CBD = \angle ABD = \angle BAC = x$.

$\therefore \Delta BCD \sim \Delta ACB$ (A.A.A.)

$$\frac{BC}{CD} = \frac{AC}{CB} \quad \text{(corr. sides, ~}\Delta\text{s)}$$

$$\frac{a}{CD} = \frac{b}{a}$$

$$CD = \frac{a^2}{b}$$

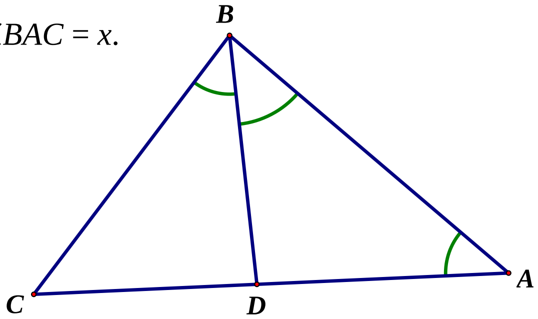


Figure 13.1

Also by sine formula, in ΔABD, $\dfrac{DA}{\sin x} = \dfrac{c}{\sin(180° - 2x)}$ … (1)

in ΔBCD, $\dfrac{CD}{\sin x} = \dfrac{a}{\sin 2x}$ … (2)

$\dfrac{(1)}{(2)}$ $\quad \dfrac{DA}{CD} = \dfrac{c}{a}$

$$\frac{DA + CD}{CD} = \frac{a + c}{a}$$

$$\frac{b}{CD} = \frac{a + c}{a}$$

$$CD = \frac{ab}{a + c}$$

$$\therefore \quad \frac{a^2}{b} = \frac{ab}{a + c}$$

$$b^2 = a(a + c)$$

Method 2

Produce *CB* to *D* such that *BD* = *AB*. Join *AD*.

$\angle BDC = \angle BAD$ (base ∠s, isos. Δ)

$= \frac{1}{2}\angle ABC$ (ext. ∠ of Δ)

$= \angle BAC$

Then Δ*ABC* ~ Δ*DAC* (A.A.A.)

$\therefore \quad \frac{AC}{BC} = \frac{DC}{AC}$ (corr. sides, ~Δs)

$\frac{b}{a} = \frac{a+c}{b}$

$b^2 = a(a + c)$

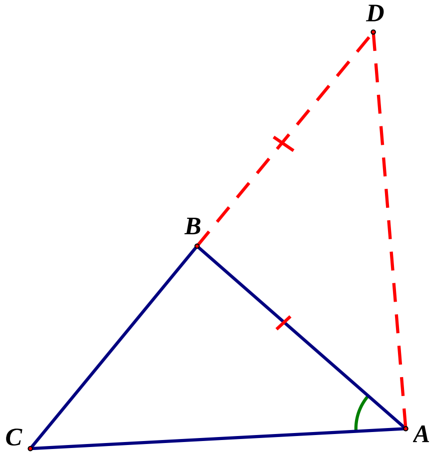


Figure 13.2

Method 3

Produce *AB* to *D* such that *BD* =*BC*.

Join *CD*.

Then $\angle BDC = \angle BCD$ (base ∠s, isos. Δ)

$= \frac{1}{2}\angle ABC$ (ext. ∠ of Δ)

$\therefore \angle CAD = \angle BDC = \angle CDA$

$\therefore \quad AC = CD = b$ (sides opp. equal ∠s)

And Δ*BCD* ~ Δ*CAD* (A.A.A.)

$\therefore \quad \frac{AC}{CB} = \frac{AD}{CD}$ (corr. sides, ~Δs)

$\frac{b}{a} = \frac{a+c}{b}$

$b^2 = a(a + c)$

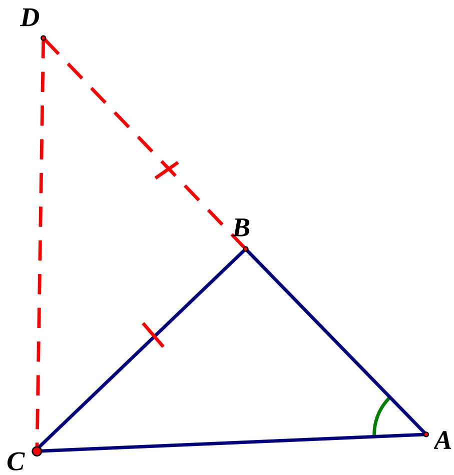


Figure 13.3

Method 4

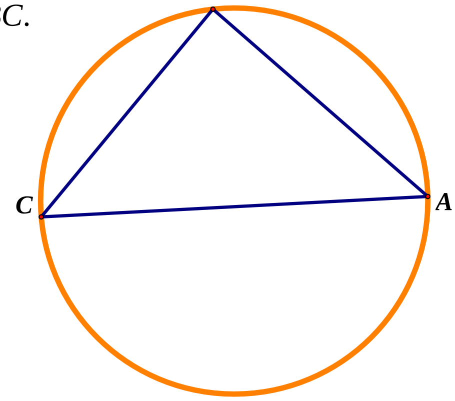


Figure 13.4

Let R be the radius of the circumscribed circle of ΔABC.

Let $\angle BAC = x$.

By sine formula,

$$\frac{a}{\sin x} = \frac{b}{\sin 2x} = \frac{c}{\sin(\pi - 3x)} = 2R$$

Then $a(a + c) = 2R \sin x\,(2R \sin x + 2R \sin 3x)$

$= 4R^2 \sin x\,(\sin x + \sin 3x)$

$= 4R^2 \sin x \times 2\sin 2x \cos x$

$= 4R^2 \sin^2 2x$

$= b^2$

Question 21

Prove that $23^n - 19^n$ is divisible by 4 for all positive integer *n*.

Method 1: By Mathematical Induction.

Let *S*(*n*) be the statement: "$23^n - 19^n$ is divisible by 4", where *n* is positive integer.

When $n = 1$, $23^1 - 19^1 = 4$, which is divisible by 4.

Therefore, *S*(1) is true.

Assume S(*k*) is true, where *k* is some positive integer.

That is, $23^k - 19^k$ is divisible by 4,

We can rewrite $23^k - 19^k = 4M$ for some positive integer *M*.

When $n = k+1$, $23^{k+1} - 19^{k+1}$

$$= 23(23^k) - 19(19^k)$$

$$= 23(23^k - 19^k) + 4(19^k)$$

$$= 23(4M) + 4(19^k)$$

$$= 4(23M + 19^k) \quad \text{which is divisible by 4}$$

Therefore, *S*(*k* +1) is also true if *S*(*k*) is true.

By Principle of Mathematical Induction, *S*(*n*) is true for all positive integer *n*.

Method 2: Using Binomial theorem.

For positive integer *n*,

$$(x+y)^n = \sum_{r=0}^{n} C_r^n x^{n-r} y^r = x^n + C_1^n x^{n-1} y + C_2^n x^{n-2} y^2 + C_3^n x^{n-3} y^3 + C_4^n x^{n-4} y^4 + \ldots + C_n^n y^n$$

By putting $x = 19$ and $y = 4$, we get:

$$23^n$$

$$= (19+4)^n$$

$$= 19^n + C_1^n 19^{n-1}\left(4\right) + C_2^n 19^{n-2}\left(4^2\right) + \ldots + C_n^n\left(4^n\right) = 19^n + \sum_{r=1}^{n} C_r^n 19^{n-r}\left(4^r\right)$$

$$= 19^n + 4\left[\sum_{r=0}^{n-1} C_{r+1}^n 19^{n-(r+1)}\left(4^r\right)\right]$$

$$\therefore \quad 23^n - 19^n = 19^n + 4\left[\sum_{r=0}^{n-1} C_{r+1}^n 19^{n-(r+1)}\left(4^r\right)\right] - 19^n = 4\left[\sum_{r=0}^{n-1} C_{r+1}^n 19^{n-(r+1)}\left(4^r\right)\right]$$

As C_r^n, 19^r and 4^r are all positive integers, $\sum_{r=0}^{n-1} C_{r+1}^n 19^{n-(r+1)}\left(4^r\right)$ is also a positive integer.

Therefore, $23^n - 19^n$ is a multiple of 4.

Method 3: By factorization

First note that $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + 1) = (x-1)\left(\sum_{r=0}^{n-1} x^r\right)$

Replace x by $\frac{x}{y}$.

$$\left(\frac{x}{y}\right)^n - 1 = \left(\frac{x}{y} - 1\right)\left[\left(\frac{x}{y}\right)^{n-1} + \left(\frac{x}{y}\right)^{n-2} + \ldots + \left(\frac{x}{y}\right) + 1\right]$$

$$y^n\left[\left(\frac{x}{y}\right)^n - 1\right] = y^n\left(\frac{x}{y} - 1\right)\left[\left(\frac{x}{y}\right)^{n-1} + \left(\frac{x}{y}\right)^{n-2} + \ldots + \left(\frac{x}{y}\right) + 1\right]$$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$$

By putting x = 23 and y =19, we get

$$23^n - 19^n = (23-19)\left[23^{n-1} + (23^{n-2})(19) + (23^{n-3})(19^2) + \ldots + (23)(19^{n-2}) + 19^{n-1}\right]$$

$$= 4\left[\sum_{r=0}^{n-1}(23^{n-1-r})(19^r)\right]$$

Note that $\sum_{r=0}^{n-1}(23^{n-1-r})(19^r)$ is a positive integer.

Therefore, $23^n - 19^n$ is a multiple of 4.

Question 22

Evaluate $\int \frac{x^3}{\left(1+x^2\right)^2}\, dx$.

Method 1

$$\int \frac{x^3}{\left(1+x^2\right)^2}\, dx = \frac{1}{2}\int \frac{x^2}{\left(1+x^2\right)^2}\, d\left(1+x^2\right)$$

$$= \frac{1}{2}\int \frac{\left(1+x^2\right)-1}{\left(1+x^2\right)^2}\, d\left(1+x^2\right)$$

$$= \frac{1}{2}\left[\int \frac{1}{1+x^2} - \frac{1}{\left(1+x^2\right)^2}\right] d\left(1+x^2\right)$$

$$= \frac{1}{2}\ln(1+x^2) + \frac{1}{2\left(1+x^2\right)} + C$$ where C is an arbitrary constant.

Method 2

Put $x = \tan u$. $dx = \sec^2 u du$.

$$\int \frac{x^3}{\left(1+x^2\right)^2}\, dx = \int \frac{\tan^3 u}{\left(\sec^2 u\right)^2} \sec^2 u\, du$$

$$= \int \frac{\sin^3 u}{\cos u}\, du$$

$$= -\int \frac{1-\cos^2 u}{\cos u}\, d(\cos u)$$

$$= -\ln|\cos u| + \frac{\cos^2 u}{2} + C$$

$$= \frac{1}{2}\ln|\sec^2 u| + \frac{1}{2\sec^2 u} + C$$

$$= \frac{1}{2}\ln|1+x^2| + \frac{1}{2(1+x^2)} + C$$ where C is an arbitrary constant.

Method 3 Resolve the integrand into partial fractions.

Let $\dfrac{x^3}{\left(1+x^2\right)^2} = \dfrac{ax+b}{1+x^2}+\dfrac{cx+d}{\left(1+x^2\right)^2}$ for some constants *a*, *b*, *c* and *d*.

$$x^3 = \left(ax+b\right)\left(1+x^2\right)+cx+d$$

Then by comparing coefficients of corresponding terms on both sides, we get

$a = 1$, $b = d = 0$, $c = -1$

$$\int \frac{x^3}{\left(1+x^2\right)^2}\,dx = \int\left[\frac{x}{1+x^2}-\frac{x}{\left(1+x^2\right)^2}\right]dx$$

$$= \frac{1}{2}\int\left[\frac{1}{1+x^2}-\frac{1}{\left(1+x^2\right)^2}\right]d\left(1+x^2\right)$$

$$= \frac{1}{2}\ln(1+x^2)+\frac{1}{2\left(1+x^2\right)}+C$$ where C is an arbitrary constant.

Question 23

Find $\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$.

Method 1

$$\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx = \int \frac{d(\cos x + \sin x)}{\cos x + \sin x}$$

$$= \ln\left|\cos x + \sin x\right| + C$$ where C is an arbitrary constant.

Method 2

$$\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx = \int \frac{\left(\cos x - \sin x\right)^2}{\cos^2 x - \sin^2 x}\, dx$$

$$= \int \frac{1 - \sin 2x}{\cos 2x}\, dx$$

$$= \int \sec 2x\, dx + \frac{1}{2}\int \frac{d\cos 2x}{\cos 2x}$$

$$= \frac{1}{2}\ln\left|\sec 2x + \tan 2x\right| + \frac{1}{2}\ln\left|\cos 2x\right| + C$$ where C is an arbitrary constant.

Method 3

$$\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx = \int \frac{1 - \tan x}{1 + \tan x}\, dx$$

$$= \int \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\, dx$$

$$= \int \tan\left(\frac{\pi}{4} - x\right) dx$$

$$= -\ln\left|\sec\left(\frac{\pi}{4} - x\right)\right| + C$$

$$= \ln\left|\cos\left(\frac{\pi}{4} - x\right)\right| + C$$ where C is an arbitrary constant.

Question 24

Show that the distance between a point $A\left(x_0, y_0\right)$ and a line L : $ax + by + c = 0$ is

$$\left|\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\right|.$$

Proof 1

Step1: Find out the distance between the origin (0, 0) and the line L.

The equation of the line L' perpendicular to L and passing through the origin is $bx - ay = 0$. By solving the equation directly, we can show that the intersection of L and L' is $P\left(\frac{-ac}{a^2+b^2}, \frac{-bc}{a^2+b^2}\right)$.

The distance between the origin and L

$$= OP = \sqrt{\left(\frac{-ac}{a^2+b^2}\right)^2 + \left(\frac{-bc}{a^2+b^2}\right)^2} = \left|\frac{c}{\sqrt{a^2+b^2}}\right|.$$

Step 2: Find out the distance between A and L by translating the origin to A.

Let $x' = x - x_0$ and $y' = y - y_0$. Referring to the coordinate system $\left(x', y'\right)$, A is the origin and the equation of line L is $a\left(x' + x_0\right) + b\left(y' + y_0\right) + c = 0$,

i.e. $ax' + by' + \left(ax_0 + by_0 + c\right) = 0$.

$\therefore$ by the result of step 1, as the distance between A and L remains unchanged after the translation, its value is given by $\left|\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\right|$.

Proof 2

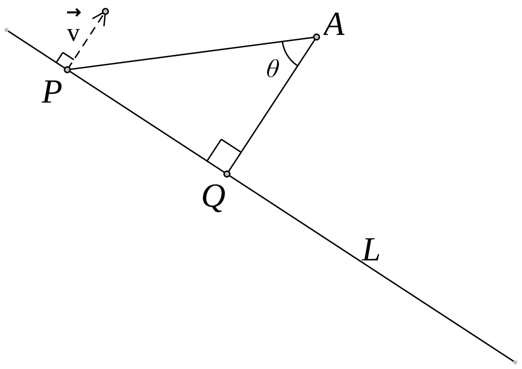


Figure 14

Suppose $P(x, y)$ is a point on the line L and θ is the angle between AP and AQ.

Then $\overrightarrow{AP} = (x - x_0)\vec{i} + (y - y_0)\vec{j}$ and let $\vec{v} = a\vec{i} + b\vec{j}$ be a vector perpendicular to line L as shown in Figure 14.

Distance between A and line L $= AQ$

$$= \left| \left|\overrightarrow{AP}\right| \cos\theta \right|$$

$$= \frac{\left| \left|\overrightarrow{AP}\right| \left|\vec{v}\right| \cos\theta \right|}{\left|\vec{v}\right|}$$

$$= \frac{\left| \left|\overrightarrow{AP}\right| \left|\vec{v}\right| \cos(\pi - \theta) \right|}{\left|\vec{v}\right|}$$

$$= \frac{\left| \overrightarrow{AP} \cdot \vec{v} \right|}{\left|\vec{v}\right|}$$

$$= \frac{\left| a(x - x_0) + b(y - y_0) \right|}{\sqrt{a^2 + b^2}}$$

$$= \frac{\left| ax + by - (ax_0 + by_0) \right|}{\sqrt{a^2 + b^2}}$$

$$= \frac{\left| -c - (ax_0 + by_0) \right|}{\sqrt{a^2 + b^2}}$$

$$= \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|$$

Question 25 (Inequality of arithmetic and geometric mean)

Given that a_1, a_2, … , a_n are non-negative numbers. Show that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n}$$

and the equality holds if and only if $a_1 = a_2 = ... = a_n$.

Note: If there are some of a_1, a_2, … , a_n are zero, the inequality obviously holds. Hence we only need to consider a_1, a_2, … , a_n all being positive.

Proof 1

Step 1: First we need to prove the following result.

Suppose x_1, x_2, … , x_n *are positive numbers such that* $x_1\ x_2 ... x_n = 1$.
Then

$$x_1 + x_2 + \cdots + x_n \ge n$$

and the equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$.

When n =2, $\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \ge 0 \ \Rightarrow\ x_1 + x_2 \ge 2\sqrt{x_1 x_2} = 2$

and the equality holds iff $x_1 = x_2 = 1$.

$\therefore$ the result holds for n = 2.

Assume the result holds for some positive integer k.
i.e. Given x_1, x_2, … , x_k are positive numbers such that $x_1 x_2 \cdots x_k = 1$.

Then $x_1 + x_2 + \cdots + x_k \ge k$, and the equality holds if and only if

$x_1 = x_2 = \cdots = x_k = 1$.

When $n = k + 1$,
let x_1, x_2, … , x_k, x_{k+1} be positive numbers such that $x_1 x_2 \cdots x_k x_{k+1} = 1$.

Since $x_1\ x_2 \ldots x_k x_{k+1} = 1$, there exists distinct positive numbers i, j with

$1 \le i,\ j \le k+1$ such that $x_i \ge 1$ and $x_j \le 1$.

W.L.O.G., say $x_k \ge 1$ and $x_{k+1} \le 1$.
By inductive hypothesis, we have $x_1 + x_2 + \cdots + x_k x_{k+1} \ge k$

Thus $x_1 + x_2 + \cdots + x_k + x_{k+1} \ge k + 1 + \left(x_k + x_{k+1} - x_k x_{k+1} - 1\right)$

where $x_k + x_{k+1} - x_k x_{k+1} - 1 = (x_k - 1)(1 - x_{k+1}) \geq 0$.

$\therefore\ x_1 + x_2 + \cdots + x_k + x_{k+1} \geq k + 1$

The equality holds iff $x_1 = x_2 = \cdots = x_{k-1} = x_k x_{k+1} = 1$ and $x_k = x_{k+1} = 1$, i.e.

$x_1 = x_2 = \cdots = x_{k-1} = x_k = x_{k+1} = 1$.

By the principle of mathematical induction, the result holds.

Step 2:

Substitute $x_i = \dfrac{a_i}{\sqrt[n]{a_1 a_2 \cdots a_n}}$ for $1 \leq i \leq n$. Then $x_1 x_2 \cdots x_n = 1$ and we have

$$\frac{a_1 + a_2 + \cdots a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} \geq n$$

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

The equality holds if and only if $\dfrac{a_i}{\sqrt[n]{a_1 a_2 \cdots a_n}} = 1$ for $1 \leq i \leq n$.

i.e. $a_1 = a_2 = \ldots = a_n$.

Proof 2:

First of all ,we need to establish the following result:

Suppose $f : I \to R$ *a twice differentiable function where I is an interval such that*

$f''(x) \leq 0$ for any $x \in I$.

Then for any positive integer n $\geq$ 2, *if* $x_1,\ x_2,\ x_2,\ \cdots,\ x_n \in I$ and

$\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \in (0,1)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, we have

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n).$$

When n = 2, suppose $x_1,\ x_2 \in I$ and $\alpha_1,\ \alpha_2 \in (0,1)$ such that $\alpha_1 + \alpha_2 = 1$.

If $x_1 = x_2$, the equality holds automatically.

So consider $x_1 < x_2$. By **Mean-Value Theorem**,

there exist $\varepsilon_1 \in \left(x_1, \alpha_1 x_1 + \alpha_2 x_2\right)$ and $\varepsilon_2 \in \left(\alpha_1 x_1 + \alpha_2 x_2, x_2\right)$ such that

1. $f\left(\alpha_1 x_1 + \alpha_2 x_2\right) - f(x_1) = \left[(\alpha_1 - 1)x_1 + \alpha_2 x_2\right] f'(\varepsilon_1) = \alpha_2\left(x_2 - x_1\right) f'(\varepsilon_1)$
2. $f\left(\alpha_1 x_1 + \alpha_2 x_2\right) - f(x_2) = \left[\alpha_1 x_1 + (\alpha_2 - 1)x_2\right] f'(\varepsilon_2) = -\alpha_1\left(x_2 - x_1\right) f'(\varepsilon_2)$

Multiply the first one by α_1 and the second one by α_2 and sum up them. We get

$$\alpha_1 f\left(\alpha_1 x_1 + \alpha_2 x_2\right) - \alpha_1 f(x_1) + \alpha_2 f\left(\alpha_1 x_1 + \alpha_2 x_2\right) - \alpha_2 f(x_2)$$
$$= \alpha_1\alpha_2\left(x_2 - x_1\right)\left[f'(\varepsilon_1) - f'(\varepsilon_2)\right]$$
$$\geq 0 \qquad (\because f''(x) \leq 0)$$
$$\therefore \alpha_1 f\left(\alpha_1 x_1 + \alpha_2 x_2\right) + \alpha_2 f\left(\alpha_1 x_1 + \alpha_2 x_2\right) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$
$$\therefore \qquad f\left(\alpha_1 x_1 + \alpha_2 x_2\right) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

Therefore, the result holds when n = 2.

Assume the result holds for some positive integer k.

i.e. If $x_1, x_2, \cdots, x_k \in I$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in (0,1)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$,

we have $f\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k\right) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_k f(x_k)$.

When $n = k + 1$, suppose $x_1, x_2, \cdots, x_k, x_{k+1} \in I$ and $\alpha_1, \alpha_2, \cdots, \alpha_k,\ \alpha_{k+1} \in (0,1)$

such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k + \alpha_{k+1} = 1$.

$$f\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1}\right)$$
$$= f\left(\alpha_1 x_1 + (1-\alpha_1)\left(\frac{\alpha_2}{1-\alpha_1}x_2 + \cdots + \frac{\alpha_k}{1-\alpha_1}x_k + \frac{\alpha_{k+1}}{1-\alpha_1}x_{k+1}\right)\right)$$
$$\geq \alpha_1 f(x_1) + (1-\alpha_1) f\left(\frac{\alpha_2}{1-\alpha_1}x_2 + \cdots + \frac{\alpha_k}{1-\alpha_1}x_k + \frac{\alpha_{k+1}}{1-\alpha_1}x_{k+1}\right)$$

(by the case of n = 2. Be careful! You should think about why $\left(\frac{\alpha_2}{1-\alpha_1}x_2 + \cdots + \frac{\alpha_k}{1-\alpha_1}x_k + \frac{\alpha_{k+1}}{1-\alpha_1}x_{k+1}\right) \in I$.)

$$= \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_k f(x_k) + \alpha_{k+1} f(x_{k+1})$$ (by the inductive hypothesis)

By the principle of mathematical induction, the result holds.

Remark: Actually, we have proved an inequality called **Jensen's Inequality.** Think about what will happen if $f''(x) \geq 0$ for any $x \in I$.

Step 2:

Consider $f(x) = \ln x$ and $I = (0, +\infty)$.

$f''(x) = -\frac{1}{x^2} < 0$ for any $x \in (0, +\infty)$

By previous result, we have

$$\ln(\alpha_1 a_1 + \alpha_2 a_2 \cdots + \alpha_n a_n) \geq \alpha_1 \ln a_1 + \alpha_2 \ln a_2 + \cdots + \alpha_n \ln a_n$$

$$\ln(\alpha_1 a_1 + \alpha_2 a_2 \cdots + \alpha_n a_n) \geq \ln(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n})$$

$$\alpha_1 a_1 + \alpha_2 a_2 \cdots + \alpha_n a_n \geq a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$$ (∵ $\ln x$ is a strictly increasing function)

By putting $\alpha_i = \frac{1}{n}$ for any $1 \leq i \leq n$, we have

$$\frac{1}{n}(a_1 + a_2 \cdots + a_n) \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

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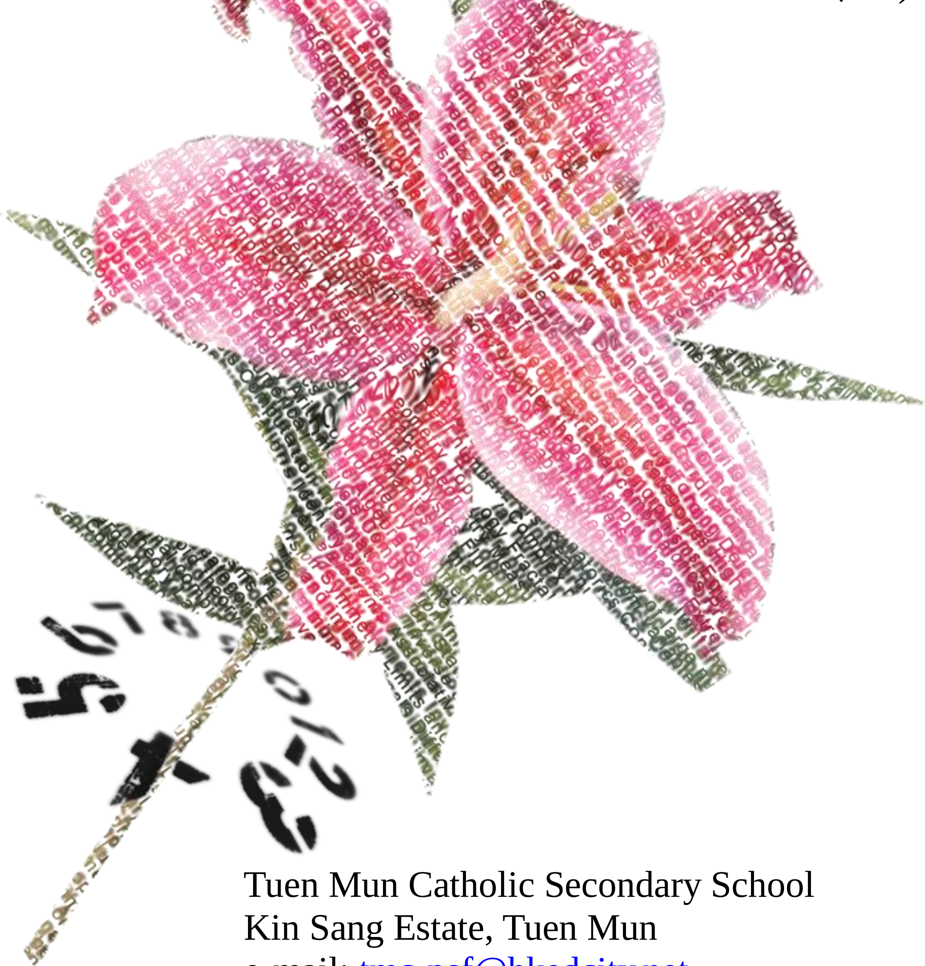

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